An alternative analytical approach to solving dynamic stochastic general equilibrium models with endogenous growth

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Abstract

Campbell (1994) argues that a clear understanding of dynamic stochastic general equilibrium models can best be achieved by an analytical approach, and he suggests loglinearizing the Euler equation and the capital accumulation equation. Inspired partly by his approach and partly by the results of Long and Plosser (1983), this paper suggests an alternative approximate analytical solution procedure to dynamic stochastic models by loglinearizing the utility function and the capital accumulation equation. This approximate solution has a simple analytical form; thus, it is relatively easy to obtain further results based on the solution. Comparing its performance with that of the conventional method, it is further argued that the suggested solution procedure is most useful for dynamic stochastic models exhibiting endogenous growth.

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1. INTRODUCTION

Macroeconomists often have to deal with the problem of solving nonlinear dynamic models, either with or without stochastic elements. A typical example is the class of dynamic stochastic general equilibrium models. After obtaining the system of dynamic equations characterizing the equilibrium, one faces the problem of 'solving' the dynamic system (in the sense of expressing the endogenous variables in terms of exogenous variables and lagged endogenous variables), before being able to draw conclusions regarding the behavior of variables in the model.

While the above steps are conceptually simple, the practical problem of this procedure is that the dynamic system is nonlinear in most cases. For example, even in the simplest Ramsey-Cass-Koopmans model, there is a mixture of additive and multiplicative elements (Campbell, 1994; Romer, 2001). This makes it impossible to obtain an exact analytical solution in general. The only well-known exception is the situation in which intertemporal elasticity of substitution is one, and capital depreciates completely in one period (Long and Plosser, 1983). Unfortunately, the usefulness of this result is limited, since the assumption of complete depreciation of capital in one period is extremely unrealistic.

Since an exact analytical solution is unavailable and since computational methods exist and are no longer costly (see, for example, Taylor and Uhlig, 1990; Judd, 1998), a lot of dynamic stochastic general equilibrium models (such as the various papers collected in Cooley, 1995) rely on the *computational* approach to obtain the equilibrium solution. In a widely cited paper, Campbell (1994) argues that compared with the computational approach, an *analytical* approach to the study of dynamic stochastic general equilibrium models would provide more insights about the dynamic effects of the underlying economic shocks. Following Baxter (1991), he focuses on the Euler equation and the capital accumulation equation. To compensate for the fact that an exact analytical solution cannot be obtained for more general cases, he derives an *approximate* analytical solution by taking a first-order Taylor approximation in logs of the variables for these two equations.[1,2] While his approach simplifies the analysis and thus achieves some of the objectives stated in his paper, his analysis still has to rely primarily on numerical calculations because the solution to a quadratic equation (such as Eq. (26) of his paper) involves a square root term and cannot be further simplified.

This paper further pursues the idea suggested in Campbell (1994). Inspired partly by his approach and partly by the results of Long and Plosser (1983), this paper suggests an alternative approximate analytical solution procedure to dynamic stochastic general equilibrium models with constant intertemporal elasticity of substitution (CIES) utility function (and with the rate of depreciation of capital in a period equals to any number between 0 and 1). The approximate solution is obtained by taking loglinear approximation of the utility function and the capital accumulation equation around the balanced growth path. (The rationale behind this proposed procedure is given in Section 3.) Since the *endogenously determined* saving rates turn out to be time-invariant, the solution has a simple analytical form. As a result, it is relatively easy to obtain further analytical results based on this solution.

To assess the performance of this alternative analytical approach, it would be useful to compare it with other possible methods. Based on a comparison of the method

[1]Campbell (1994, p. 465) also explains why, in a homoskedastic setting, this approximate solution is the same as that based on taking a second-order Taylor approximation in logs of the variables (such as in Christiano, 1988).

[2]This method of solving nonlinear dynamic discrete-time models has been generalized in Uhlig (1999), who solves matrix quadratic equations after loglinerarizing the necessary equations characterizing the equilibrium. Note that while Uhlig generalizes the approach used in King et al. (1988) and Campbell (1994), he has reservation regarding the term 'analytical approach' used in Campbell (1994); see Uhlig (1999, footnote 3).

suggested in Campbell (1994) and that suggested in this paper, as well as on the idea drawn from existing literature on theoretical growth models, it is argued that the solution procedure suggested in this paper is most useful for those dynamic stochastic models in which sustained growth can be generated endogenously (such as Romer 1986; Rebelo, 1991; Collard, 1999).

The analysis presented in this paper also suggests that the assumption of a nonstandard capital accumulation equation used in a number of papers may not be necessary. This assumption has been used in Hercowitz and Sampson (1991), and then followed by a number of other papers such as Kocherlakota and Yi (1995, 1997) and Collard (1999). These papers assume that the current level of capital stock is a Cobb-Douglas function of its lag term and current investment. It appears that the sole reason for using this form of capital accumulation (instead of the standard linear form) is to derive a closed-form solution. The analysis offered in this paper shows that as long as the Cobb-Douglas capital accumulation equation is interpreted as the loglinear approximation of the linear capital accumulation equation around the balanced growth path, the approach of Hercowitz and Sampson (1991) can easily be reconciled with the standard approach.

The remaining parts of this paper are organized as follows. To make a comparison between the results of the procedure offered here and existing results, and to systematically show how the suggested approximate solution is developed, this paper first focuses on the basic nonstochastic neoclassical growth model (with fixed labor supply), which has been examined in Section 3.2.1 of King et al. (1988), and Section 2 of Campbell (1994). Section 2 of this paper follows Campbell (1994) and loglinearizes the Euler equation and the capital accumulation equation. Section 3 suggests a different solution procedure by taking the loglinear approximation of the utility function and the capital accumulation equation. (For convenience, this method will be called the '*alternative*' method, and the method suggested in Section 2 will be called the '*conventional*' method.) Section 4 compares the two approximate solutions, and suggests the conditions under which the two methods will produce similar equations of motion for the variables.

Partly based on the analysis in Section 4, it is argued that the method suggested in this paper is most useful for the class of dynamic stochastic general equilibrium models with endogenous growth. To demonstrate this point, Section 5 applies the proposed solution procedure to a real-business-cycle model with endogenous growth examined in Hercowitz and Sampson (1991), except that the Cobb-Douglas capital accumulation equation used in that paper is replaced by the standard specification. Section 6 provides concluding remarks.

2. THE CONVENTIONAL LOGLINEAR APPROXIMATE SOLUTION TO THE RAMSEY-CASS-KOOPMANS MODEL

The closed economy is populated by a constant and large number of identical agents (consumer-producers). The production technology is represented by a standard Cobb-Douglas production function:

$$Y_t = AF\left(K_t, (1+\gamma_x)^t N_t\right) = AK_t^{\alpha}\left[(1+\gamma_x)^t N_t\right]^{1-\alpha} = A(1+\gamma_x)^{(1-\alpha)t} K_t^{\alpha}, \quad (1)$$

where $0 < \alpha < 1$; γ_x ($\gamma_x \geq 0$) is the rate of exogenous labor-augmenting technological progress; the (inelastically supplied) labor input at each period, N_t, has been normalized to be 1; and Y_t and K_t are, respectively, the output and the associated capital input at period t. (As labor input has been normalized to 1, Y_t and K_t can also be interpreted respectively as output per worker and capital per worker.)

On the preference side, the representative agent chooses a consumption path to maximize the following lifetime utility function:

$$\sum_{j=0}^{\infty} \beta^j U(C_{t+j}) = \sum_{j=0}^{\infty} \beta^j \frac{C_{t+j}^{\left(1-\frac{1}{\sigma}\right)} - 1}{1-\frac{1}{\sigma}}, \quad (2)$$

where β is the subjective time discount factor, σ ($\sigma > 0$) is the intertemporal elasticity of substitution, and C_t is consumption at period t.

The capital accumulation process is described by:

$$K_{t+1} = (1-\delta) K_t + I_t, \tag{3}$$

where δ ($0 \leq \delta \leq 1$) is the depreciation rate per period, and I_t is investment at period t. Finally, the model is closed by the resource constraint:

$$Y_t = C_t + I_t. \tag{4}$$

The use of Cobb-Douglas production function (1) and CIES utility function (2) is quite standard in the literature. Moreover, the analysis of King et al. (1987, Sections A3 and A4; 1988, Section 2.3) suggest that such restrictions on technology and preferences would lead to the phenomenon of balanced growth, in which certain key variables grow at constant but possibly different rates. It is well known that along the balanced growth path of the Ramsey-Cass-Koopmans model, output, consumption, investment, and capital all grow at γ_x, the rate of exogenous technological progress.

To transform a growing economy into a 'no-growth' economy, define the variable per unit of effective labor as the variable per worker divided by $(1+\gamma_x)^t$, and denote it in a lowercase letter (such as $c_t = C_t / \left(1+\gamma_x\right)^t$). With this transformation, (3) leads to:

$$(1+\gamma_x) k_{t+1} = (1-\delta) k_t + i_t, \tag{5}$$

and (1) and (4) lead to:

$$A k_t^{\alpha} = c_t + i_t. \tag{6}$$

Moreover, it can be shown that the representative agent is maximizing an equivalent utility function defined in terms of consumption per unit of effective labor:

$$\sum_{j=0}^{\infty} (\beta_x)^j \frac{c_{t+j}^{\left(1-\frac{1}{\sigma}\right)} - 1}{1-\frac{1}{\sigma}}, \tag{7}$$

where the effective discount factor β_x differs from β owing to the transformation of the preference specification (see, for example, Section 2.4 of King et al., 1988) and is given by:

$$\beta_x = \beta (1 + \gamma_x)^{1-\frac{1}{\sigma}}. \tag{8}$$

To guarantee finiteness of lifetime utility, the restriction $\beta_x < 1$ is required. Note that when $\sigma = 1$, $\beta_x = \beta$ according to (8); moreover, the utility function $\frac{C_{t+j}^{\left(1-\frac{1}{\sigma}\right)}-1}{1-\frac{1}{\sigma}}$ in (2) becomes $\ln C_{t+j}$, and the corresponding utility function in (7) becomes $\ln c_{t+j}$.

The first-order condition for optimal choice of this model, given the objective function (7) and the constraints (5) and (6) is

$$\beta_x c_{t+1}^{-\frac{1}{\sigma}} \left[1 - \delta + \alpha A k_{t+1}^{-(1-\alpha)}\right] = (1 + \gamma_x)\, c_t^{-\frac{1}{\sigma}}. \tag{9}$$

Moreover, (5) and (6) lead to:

$$(1 + \gamma_x)\, k_{t+1} = (1 - \delta)\, k_t + A k_t^{\alpha} - c_t. \tag{10}$$

Therefore, the optimal solution of the economy, in terms of consumption and capital, is described by the system of two *nonlinear* difference equations: (9) and (10).[3] The optimal choice also satisfies a transversality condition.

An exact analytical solution is not possible for the above system of optimal choice, except for the special case in which $\sigma = 1$ and $\delta = 1$ (Long and Plosser, 1983; McCallum, 1989). For the general case, Campbell (1994) discusses the advantages of an analytical approach and suggests loglinearizing the Euler equation and the capital accumulation equation.

The analysis in this section follows this idea but modifies the method slightly. In particular, the following analysis expresses the approximate solution in terms of

[3]To economize on the use of words, consumption per worker, consumption per effective unit of labor, deviation of consumption per effective unit of labor from the steady-state value, and so forth are simply represented by the term 'consumption' in this paper, if the context is clear.

fundamental preference and technology parameters, instead of the real interest rate used in Campbell (1994). While expressing the approximate solution in terms of the real interest rate is useful for numerical calculations (as is done in Section 2.6 of Campbell (1994) and, indirectly, in Section 4 of this paper), expressing the solution in terms of fundamental parameters facilitates the comparison of the conventional and alternative approximate solutions.

To obtain the loglinear approximation, first use the Euler equation and the capital accumulation equation to get the steady-state values of k and c. Equations (9) and (10) imply that k and c are related by:

$$1 - \delta + \alpha A k^{-(1-\alpha)} = \frac{1+\gamma_x}{\beta_x}, \tag{9a}$$

and

$$c = \left[\frac{1+\gamma_x - \beta_x (1-\delta)}{\beta_x \alpha} - (\delta + \gamma_x)\right] k. \tag{10a}$$

Next, linearize (9) and (10) around $\ln k$ and $\ln c$. Linearizing (9) leads to:

$$(\ln c_{t+1} - \ln c) + \theta_{ck} (\ln k_{t+1} - \ln k) = (\ln c_t - \ln c), \tag{11}$$

where

$$\theta_{ck} = \sigma (1-\alpha) [1 - \beta_x (1 - \delta_x)], \tag{11a}$$

and δ_x, which can be interpreted as the *effective depreciation rate* for the transformed problem, is defined as:

$$\delta_x = \frac{\delta + \gamma_x}{1+\gamma_x}. \tag{12}$$

It is easy to see that $0 \leq \delta_x \leq 1$. Similarly, linearizing (10) leads to:

$$(\ln k_{t+1} - \ln k) = \theta_{kk} (\ln k_t - \ln k) - \theta_{kc} (\ln c_t - \ln c), \tag{13}$$

where

$$\theta_{kk} = \frac{1}{\beta_x}, \tag{13a}$$

and

$$\theta_{kc} = \frac{1 - \beta_x \left[1 - \delta_x \left(1 - \alpha\right)\right]}{\beta_x \alpha}. \tag{13b}$$

Eqs. (11) and (13) form a system of *loglinear* difference equations in consumption and capital stock. This system is an *approximation* of the nonlinear system of difference equations (9) and (10).

There are many ways to solve the above loglinear system. In the following analysis, the method of undetermined coefficients is used, and the emphasis is on the elasticity of capital stock with respect to its lag, η_{kk}, defined as (14) below, or in (20) of Campbell (1994). This coefficient is focused because the results in Rebelo (1991) and Lau (1999) suggest that η_{kk} equals to 1 for an endogenous growth model. (The limiting case of the Ramsey-Cass-Koopmans model with no exogenous technological progress as α tends to 1 is the AK endogenous growth model). On the other hand, it is not clear whether there are similar special results for other coefficients such as the elasticity of consumption with respect to current capital, given in (19) of Campbell (1994). Focusing on the dynamic equation of the capital stock turns out to simplify the analysis, and to suggest the conditions under which the alternative method suggested in this paper will be most useful.

As the method of undetermined coefficients is well known, I only sketch the procedure and present the main results.[4] First, transform the system of first-order difference equations in consumption and capital, (11) and (13), into a second-order difference equation in capital. Second, guess that the solution of log capital in terms of its lag is

$$\ln k_{t+1} - \ln k = \eta_{kk} \left(\ln k_t - \ln k\right), \tag{14}$$

where η_{kk} is an unknown coefficient to be determined. Substituting (14) into the second-order difference equation in capital leads to an equation with only one vari-

[4]Detailed analysis can be found in Lau (2002), who examines a more general case with the presence of stochastic productivity shocks.

able: the deviation of capital per effective unit of labor from the steady-state value. Equating the coefficients of $(\ln k_t - \ln k)$ on both sides of this equation and simplifying give the unknown coefficient η_{kk} as:[5]

$$\eta_{kk} = \frac{(1+\theta_{kk}+\theta_{kc}\theta_{ck}) - \sqrt{(1+\theta_{kk}+\theta_{kc}\theta_{ck})^2 - 4\theta_{kk}}}{2}. \tag{15}$$

Since $\sigma > 0$, $0 < \beta_x < 1$, $0 < \alpha < 1$, and $0 \leq \delta_x \leq 1$, it is easy to show from (11a), (13a) and (13b) that $\theta_{ck} > 0$, $\theta_{kk} > 0$, and $\theta_{kc} > 0$. As a result, η_{kk} in (15) is between zero and one.

3. AN ALTERNATIVE APPROXIMATE SOLUTION TO THE RAMSEY-CASS-KOOPMANS MODEL

It is well known that an analytical solution for a dynamic stochastic general equilibrium model with CIES utility function is possible only for the special case of $\sigma = 1$ (log utility) and $\delta = 1$; see, for example, Romer (2001, Chapter 4). However, while this special case is useful for some expositional purposes, it is not very helpful from a practical perspective, as the assumption of complete depreciation of capital in one period is very unrealistic. For the more general cases, Campbell (1994) mentions the advantages of an approximate analytical solution. However, most of the results in his paper are still based on numerical calculations, and few of the results can be deduced analytically. The same applies to the analysis in Section 2 of this paper. While it streamlines slightly the derivation of the conventional solution by focusing on the elasticity of capital stock with respect to its own lag, not much can be obtained without further use of numerical calculations, since the solution to a quadratic equation

[5] Equating the coefficients of $(\ln k_t - \ln k)$ gives the following quadratic equation: $\eta_{kk}^2 - (1+\theta_{kk}+\theta_{kc}\theta_{ck})\,\eta_{kk} + \theta_{kk} = 0$. It can be shown that there are two unequal positive real roots to this quadratic equation. Moreover, the larger root (which is larger than β_x^{-1}) is excluded, since the transversality condition would otherwise be violated.

(Eq. (15) above) involves a square root term and cannot be further simplified.

This section suggests an alternative approximate analytical solution to the Ramsey-Cass-Koopmans model with CIES utility function ($\sigma > 0$) when the depreciation rate is not restricted ($0 \leq \delta \leq 1$). The underlying idea of the alternative solution can be understood as follows. It is observed from the analysis in Section 2 that the dynamic optimization problem is to maximize (7) subject to (5) and (6). As mentioned in Campbell (1994) and Romer (2001), the difficulty lies in the fundamental nonlinearity arising from the *interaction between additive and multiplicative elements* in the model: multiplicative elements appear in the utility function in (7) and in the production function in the left-hand side of (6), but additive elements appear in (5) and (6). With log utility and complete depreciation in one period, the capital accumulation equation (5) and the utility function (7) become loglinear, and there is one remaining linear element appearing in (6). In this case, it turns out that the endogenously determined saving rates are time-invariant, and the time-invariant saving rates prevent the linearity in (6) from causing problems, since optimal level of investment is a constant fraction of output. Therefore, an exact analytical solution is possible for this special case.

Based on the above observation, a logical *conjecture* is that *if the utility function and the capital accumulation equation could be made loglinear, then an analytical solution may be possible.* While an *exact* analytical solution is impossible when the capital accumulation equation (5) is not loglinear for the general case of incomplete depreciation ($\delta < 1$) and the utility function (7) is not loglinear for $\sigma \neq 1$, the following analysis seeks an *approximate analytical solution* to the above problem by maximizing the *loglinear approximation of utility function* (7) subject to (6) and the *loglinear approximation of the capital accumulation equation* (5).

It is easy to see from (5) that investment per unit of effective labor and capital per unit of effective labor along the balanced growth path are related by:

$$(\delta + \gamma_x) \, k = i. \tag{5a}$$

Linearizing the capital accumulation equation (5) around $\ln k$ and $\ln i$, where k and i are related according to (5a), leads to

$$\ln k_{t+1} - \ln k = (1 - \delta_x) (\ln k_t - \ln k) + \delta_x (\ln i_t - \ln i) \, ,$$

which is equivalent to:

$$\ln k_{t+1} = (1 - \delta_x) \ln k_t + \delta_x \ln i_t - \delta_x \ln (\delta + \gamma_x) \, , \tag{5b}$$

where δ_x is defined before in (12). Note that when $\delta = 1$, (5b) is an exact loglinear form of the capital accumulation equation (5). On the other hand, the (first-order) loglinear approximation of the CIES utility function in (7) around $\ln c$ is given by:

$$\frac{c_{t+j}^{\left(1-\frac{1}{\sigma}\right)} - 1}{1 - \frac{1}{\sigma}} = \frac{c^{\left(1-\frac{1}{\sigma}\right)} - 1}{1 - \frac{1}{\sigma}} + c^{\left(1-\frac{1}{\sigma}\right)} (\ln c_{t+j} - \ln c) \, . \tag{7a}$$

The alternative approximate solution is obtained by maximizing

$$\sum_{j=0}^{\infty} (\beta_x)^j \left[c^{\left(1-\frac{1}{\sigma}\right)} \ln c_{t+j} \right] \tag{7b}$$

subject to (6) and (5b), after ignoring the constant terms in (7a). Applying the method of dynamic programming, the solution is described by a time-invariant policy function that maps the state variable(s) into the agent's control variable(s). For this problem, the state variable at period t is k_t, and the control variable is i_t. The value function, $V(k)$, is defined by the Bellman equation of optimality:

$$V(k_t) = \max_{i_t} \left[c^{\left(1-\frac{1}{\sigma}\right)} \ln (A k_t^{\alpha} - i_t) + \beta_x V(k_{t+1}) \right] , \tag{16}$$

subject to the approximate capital accumulation equation (5b).

The solution is given in Appendix A. It is shown that the optimal policy rule is expressed in terms of the underlying parameters as follows:

$$\ln i_t = \ln \left[\frac{\beta_x \alpha \delta_x A}{1 - \beta_x (1 - \delta_x)} \right] + \alpha \ln k_t. \tag{17}$$

Substituting (17) into (5b) gives the equilibrium dynamic equation of the capital stock as:

$$\ln k_{t+1} = [1 - (1 - \alpha) \delta_x] \ln k_t + \delta_x \ln \left[\frac{\beta_x \alpha A}{1 + \gamma_x - \beta_x (1 - \delta)} \right],$$

or, in deviation form (similar to Eq. (14) in Section 2),

$$\ln k_{t+1} - \ln k = [1 - (1 - \alpha) \delta_x] (\ln k_t - \ln k). \tag{18}$$

An important characteristic of the above approximate solution is that the (*endogenous*) *saving rates* are *time-invariant*, as implied by (1) and (17).

The steps of the alternative approximate analytical solution procedure suggested in this paper can be summarized as follows:

1. To loglinearize the capital accumulation equation around the balanced growth path (or the steady state if the variables have been transformed into their 'no-growth' counterparts) of the dynamic stochastic model.

2. To loglinearize the CIES utility function. If $\sigma = 1$, this step can be skipped since the utility function is already in log form.

3. To obtain the analytical solution by maximizing the (approximate) log-linear utility function subject to the (approximate) log-linear capital accumulation equation (as well as other equations of motion of the system). In particular, obtain the optimal policy functions (expressing the control variables in terms of the state variables) and the associated dynamic equations of the state variables.

4. The policy function obtained by this method is one with a constant saving rate, as indicated by the analysis in this section and in Section 5.2. As such, further analysis making use of the dynamic properties of the model is relatively easy.

4. UNDER WHAT CONDITIONS WILL THE ALTERNATIVE APPROXIMATE ANALYTICAL SOLUTION BE MOST USEFUL?

Stimulated partly by the analytical approach advocated in Campbell (1994) and partly by the solution of the special case in which the single-period utility function is logarithmic and depreciation of capital is complete in one period (Long and Plosser, 1983), Section 3 suggests an alternative approximate analytical solution to the Ramsey-Cass-Koopmans model with CIES utility function by loglinearizing the utility function and the capital accumulation equation.[6] The optimal policy rule (17) and the dynamic equation of capital (18) turn out to be analytically simpler. It is possible to derive a simple approximate analytical solution because of a property similar to that in Long and Plosser (1983): the endogenously determined saving rates are time-invariant under an (approximate) loglinear utility function and an (approximate) loglinear capital accumulation equation.

[6]To a lesser extent, the suggested approximate solution procedure is also stimulated by the linear-quadratic approach used in Kydland and Prescott (1982) and McGrattan (1990). That approach solves a problem with a quadratic objective function and linear constraints. For the time-to-build model considered in Kydland and Prescott (1982) as well as the stochastic growth model considered in McGrattan (1990), the objective function is nonlinear but the constraints are linear. Kydland and Prescott (1982, Section 4) approximate the intertemporal utility function with a quadratic function, and maximize the *approximate quadratic utility function* subject to the *original linear technological constraints*. A similar approach is used in McGrattan (1990, pp. 42-43). On the other hand, the approach used in an early version of this paper (which only considers log utility function, a special case of the CIES utility function in which $\sigma = 1$) is to maximize the *original loglinear utility function* (when $\sigma = 1$) subject to the *loglinear approximation* of the capital accumulation equation.

While this alternative approach extends the Long-Plosser result to give an approximate analytical solution to the case of more general depreciation rates and more general utility functions, it is helpful to compare this (approximate) solution to the conventional one. It turns out that this comparison also suggests the conditions under which the alternative solution procedure is most useful.

The following analysis focuses on the comparison of the elasticity of capital stock with respect to its lag in the two approximate solutions: η_{kk} in (14) for the conventional solution and $1-(1-\alpha)\,\delta_x$ in (18) for the alternative solution.

To see whether η_{kk} in (14) is close to $1-(1-\alpha)\,\delta_x$ in (18) for different parameter combinations, first note that η_{kk} depends on all the parameters β_x, σ, α and δ_x (with δ_x further depending on δ and γ_x), whereas $1-(1-\alpha)\,\delta_x$ depends only on α and δ_x but not on β_x and σ.[7] Table 1 compares η_{kk} with $1-(1-\alpha)\,\delta_x$ for a quarterly model. It is assumed that $\delta = 0.025$ (10% at an annual rate) and $\gamma_x = 0.005$ (2% at an annual rate). Moreover, β_x is chosen so as to make the steady-state real interest rate equal to 0.015 (6% at an annual rate). These values are chosen to resemble the long-run behavior of the US economy; see Campbell (1994) also. On the other hand, the exponent on capital in the production function (α) and the intertemporal elasticity of substitution (σ) may take different values in Table 1. The values of parameter α examined include 0.33, 0.42 (as used in King et al. [1988]), 0.67 (as suggested by Mankiw, Romer and Weil [1992] for a neoclassical growth model augmented with human capital), 0.8 (for a model with a broad concept of capital, in order to be consistent with the convergence rate reported in Barro and Sala-i-Martin [1992]), and 0.95.

First, it is observed from Table 1 that η_{kk} increases as α increases from 0 to 1 (and

[7] Note that for the alternative solution, the coefficient $1-(1-\alpha)\,\delta_x$ of dynamic equation (18) does not depend on σ, but the (endogenously determined and time-invariant) saving rates in (17) depend on σ (through the term β_x).

decreases with respect to σ). These results have been confirmed analytically in Lau (2002). Second, it is obvious that $1-(1-\alpha)\,\delta_x$ increases in α (and is independent of σ). The third and most important observation is that as α tends to 1, both η_{kk} and $1-(1-\alpha)\,\delta_x$ tend to 1, and thus the difference between η_{kk} and $1-(1-\alpha)\,\delta_x$ goes to 0. This pattern is observed for all values of σ. Unreported results show that the pattern in Table 1 is also robust to minor changes in γ_x, δ, and the real interest rate (or β_x).

The numerical calculations in Table 1 suggest that the two approximate analytical solutions to the Ramsey-Cass-Koopmans model (with Cobb-Douglas production function and CIES utility function) are very similar when α is close to 1 and other parameter values are chosen to resemble the long-run behavior of the US economy. What is the implication of these results?

In a literal sense, parameter α is the exponent of capital in the production function of the Ramsey-Cass-Koopmans model. According to most empirical results, the estimate of α lies between 0.30 and 0.45, and is far below 1. As a result, the conventional and alternative approximate solutions for the Ramsey-Cass-Koopmans model differ quite substantially when α lies in this range. In this sense, the alternative solution may not be particularly useful, since it is quite different from the conventional approximate solution which is known to have a reasonable degree of accuracy (Campbell, 1994, p. 465).

On the other hand, the alternative solution procedure is useful when one interprets variable k_t in the previous sections as a *broad concept of capital*, including, possibly, physical and human capital, financial capital, and stock of knowledge (Rebelo, 1991, p. 502; Sala-i-Martin, 1990, p. 5). According to this interpretation, parameter α takes the role of the sum of exponents of all accumulable inputs in the production function of the Cobb-Douglas form.[8] The production technology exhibits constant

[8]For example, the exponent of private physical capital (α) is less than 1 in Eq. (20) of Section 5,

returns to scale with respect to accumulable inputs if $\alpha = 1$ but is of diminishing returns to scale with respect to accumulable inputs if $\alpha < 1$. Moreover, in the case of $\alpha < 1$, the diminishing returns will set in more slowly if α is closer to 1.

According to the latter interpretation, the numerical results in Table 1 imply that the conventional and alternative approximate solutions will give *quantitatively similar* equations of motion when the production technology approaches *constant returns to scale with respect to accumulable inputs.* The above analysis is consistent with the known results about endogenous growth models in the literature (such as Rebelo, 1991 and Lau, 1999). The results in these papers suggest that if the equilibrium dynamic equation of log capital is of first order for an endogenous growth model (such as the stochastic or nonstochastic AK model), then the coefficient of the lagged capital must be 1 for both solutions, because this is the *restriction* imposed by the theoretical model so as to produce *sustained endogenous growth.* As a result, both conventional and alternative approximate methods will give the same solution (at least for the nonstochastic model).[9] Since the Ramsey-Cass-Koopmans model when α tends to 1 (and $\gamma_x = 0$) exhibits sustained growth endogenously and the difference equation of log capital for either solution of this model (Eq. (14) for the conventional method and Eq. (18) for the alternative method) is of first order, it is not surprising to see the similarity of the two solutions when α tends to 1.

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which is the same as Eq. (1) of Hercowitz and Sampson (1991). On the other hand, owing to the presence of knowledge (which is modelled as a by-product of accumulated investment and research activities) in (20), it can easily be seen that the production function is of constant returns to scale with respect to all accumulable inputs (K_t and H_t).

[9]For a stochastic model, the coefficients of the stochastic shocks in the equilibrium dynamic equation of capital (which is a first order stochastic difference equation) may differ for the two approximate solutions, while the coefficient of capital stock in the previous period must be the same (and equal to 1). This can be observed by comparing (36) and (39) of the model in Section 5.

5. APPLYING THE ALTERNATIVE METHOD TO A DYNAMIC STOCHASTIC MODEL WITH ENDOGENOUS GROWTH

The results for the Ramsey-Cass-Koopmans model in previous sections, together with existing results in the literature about stochastic endogenous growth models, suggest that the alternative approximate analytical solution procedure is most useful for the class of dynamic stochastic general equilibrium models with endogenous growth. To illustrate this point, this section applies the method to a real-business-cycle model with endogenous growth examined in Hercowitz and Sampson (1991).

There are two reasons in choosing the Hercowitz and Sampson (1991) model. First, a model with labor-leisure choice and stochastic elements is selected to demonstrate the applicability and simplicity of the alternative solution procedure to models with these features.[10] Second, that paper uses a nonstandard specification of the capital accumulation equation, which enables them to obtain a closed-form solution. Their approach has been followed in a number of papers such as Kocherlakota and Yi (1995, 1997) and Collard (1999). The following analysis shows that such a nonstandard assumption is not necessary. In fact, using the alternative approximate solution procedure suggested in Section 3, their Cobb-Douglas capital accumulation equation arises naturally from the loglinear approximation of the standard capital accumulation equation around the balanced growth path.

The following model is taken from Hercowitz and Sampson (1991), except that their Cobb-Douglas capital accumulation equation (Eq. (2) in their paper) is replaced by a standard specification. Since the model has been described in detail in their paper, it is discussed only briefly here. The closed economy is populated by a constant and

[10]As is demonstrated in Section 5.1, while it is possible to use the conventional method to obtain an approximate analytical solution for this model, the analysis would lead to very convoluted expressions once the labor-leisure choice and stochastic shocks are included. The simplicity of the alternative method can easily be seen after comparing the analysis in Sections 5.1 and 5.2.

large number of identical agents (consumer-producers). The representative agent's lifetime utility function is given by:

$$E_t \left[\sum_{j=0}^{\infty} \beta^j \ln \left(C_{t+j} - H_{t+j} N_{t+j}^{1+\omega} \right) \right], \tag{19}$$

where $\omega > 0$, H_t is an index of knowledge at period t, and E_t is the expectation operator conditional on the information set at period t. Note that labor input N_t is a choice variable in this model. The representative firm's production function is:

$$Y_t = AK_t^{\alpha} \left(H_t N_t \right)^{1-\alpha} \exp \left(z_{1,t} \right), \tag{20}$$

where $0 < \alpha < 1$ and $z_{1,t}$ is a productivity shock.

In Hercowitz and Sampson (1991), capital stock evolves stochastically over time according to a Cobb-Douglas specification. While Hercowitz and Sampson (1991), as well Kocherlakota and Yi (1997) and Collard (1999) use adjustment costs as the justification for adopting this nonstandard capital accumulation equation, it appears that the main reason is to obtain a closed-form solution.[11] This assumption enables these authors to derive analytically the relationships among various variables according to their models and to choose the appropriate econometric procedures. Even though this approach provides some benefits, one may wonder whether the conclusions based on this nonstandard assumption would still hold with the standard capital accumulation equation.

In the following analysis, a more standard approach is adopted. To make the differences from the original model as small as possible, the following stochastic version of the standard capital accumulation equation is used:

$$K_{t+1} = \left[\left(1 - \delta \right) K_t + I_t \right] \exp \left(z_{2,t+1} \right), \tag{21}$$

[11] In each of these papers, no closed-form solution is available if the standard capital accumulation equation is used, but a closed-form solution is obtained once a nonstandard specification is adopted.

where $z_{2,t}$ is a shock to the capital accumulation equation.

Following the idea presented in Arrow (1962) and Romer (1986), Hercowitz and Sampson (1991) assume that knowledge is a by-product of the accumulated investment and research activities in the economy. Specifically, it is assumed in their paper that:

$$H_t = \bar{K}_t, \tag{22}$$

where $\bar{K}_t$ is the average capital stock across firms.

Finally, the resource constraint of the economy is the same as Eq. (4), and each of the stochastic shock processes ($z_{1,t}$ and $z_{2,t}$) is assumed to be a stationary first-order autoregressive process as follows:

$$z_{1,t} = \phi_1 z_{1,t-1} + \varepsilon_{1,t}, \tag{23}$$

$$z_{2,t} = \phi_2 z_{2,t-1} + \varepsilon_{2,t}, \tag{24}$$

where $-1 < \phi_1 < 1$, $-1 < \phi_2 < 1$, and $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are white noise processes.

Section 5.1 obtains the approximate analytical solution by the conventional method, and Section 5.2 uses the alternative method. Section 5.3 compares these two methods.

5.1. The conventional method

The first step of the conventional approximate solution procedure is to find the first-order conditions corresponding to individual optimization (taking the aggregate variables as exogenously given). Specifically, individual agents in this economy take the time path of average capital stock as given, because the action of an individual has (approximately) no effect on per-capita capital stock under the assumption of a large number of economic agents.

Applying the method of dynamic programming to this optimization problem, the state variables at period t are K_t, $\bar{K}_t$, $z_{1,t}$ and $z_{2,t}$. On the other hand, the control variables are I_t and N_t. The dynamic optimization problem is defined by the following Bellman equation of optimality:

$$V\left(K_t, \bar{K}_t, z_{1,t}, z_{2,t}\right) =$$

$$= \max_{I_t, N_t} \left\{ \ln \left[A K_t^{\alpha} \left(\bar{K}_t N_t\right)^{1-\alpha} \exp\left(z_{1,t}\right) - I_t - \bar{K}_t N_t^{1+\omega} \right] + \beta E_t V\left(K_{t+1}, \bar{K}_{t+1}, z_{1,t+1}, z_{2,t+1}\right) \right\}, \tag{25}$$

subject to the various equations of motion (21), (23) and (24).

It can be shown (in Appendix B) that the first-order conditions are given by:

$$\frac{1}{C_t - \bar{K}_t N_t^{1+\omega}} = \beta E_t \left[\frac{1 - \delta + \alpha A K_{t+1}^{-(1-\alpha)} \left(\bar{K}_{t+1} N_{t+1}\right)^{1-\alpha} \exp\left(z_{1,t+1}\right)}{C_{t+1} - \bar{K}_{t+1} N_{t+1}^{1+\omega}} \exp\left(z_{2,t+1}\right) \right], \tag{26}$$

and

$$\frac{(1+\omega)\bar{K}_t N_t^{\omega}}{C_t - \bar{K}_t N_t^{1+\omega}} = \frac{(1-\alpha) A K_t^{\alpha} \left(\bar{K}_t N_t\right)^{1-\alpha} \exp\left(z_{1,t}\right)}{N_t} \left(\frac{1}{C_t - \bar{K}_t N_t^{1+\omega}} \right). \tag{27}$$

Equation (26), or more generally (B1) in Appendix B, is the intertemporal Euler equation, relating the marginal utility of current consumption with the expected marginal utility of consumption in the next period. Equation (27), or more generally (B2), is the intratemporal condition relating the marginal utility of consumption and marginal disutility of work at a particular period, and it represents the trade-off between consumption and leisure. Simplifying Equation (27) leads to

$$N_t = \left(\frac{1-\alpha}{1+\omega} A \right)^{\frac{1}{\alpha+\omega}} \left(K_t\right)^{\frac{\alpha}{\alpha+\omega}} \left(\bar{K}_t\right)^{\frac{-\alpha}{\alpha+\omega}} \left(\exp z_{1,t}\right)^{\frac{1}{\alpha+\omega}}. \tag{27a}$$

The second step is to specify the consistency condition between individual and aggregate variables. At equilibrium, the following equality

$$K_t = \bar{K}_t \tag{28}$$

holds, because of the assumption of identical agents.

The next step is to combine the individual optimality conditions (26) and (27a), the equilibrium condition (28) and the laws of motion (21), (23) and (24). Combining (27a), (26) and (21) with the equilibrium condition (28) lead to, respectively,

$$N_t = \left(\frac{1-\alpha}{1+\omega}A\right)^{\frac{1}{\alpha+\omega}} \left(\exp z_{1,t}\right)^{\frac{1}{\alpha+\omega}}, \tag{29}$$

$$\frac{1}{C_t - K_t N_t^{1+\omega}} = \beta E_t \left[\frac{1-\delta+\alpha A N_{t+1}^{1-\alpha}\exp\left(z_{1,t+1}\right)}{C_{t+1} - K_{t+1}N_{t+1}^{1+\omega}} \exp\left(z_{2,t+1}\right)\right], \tag{30}$$

and

$$K_{t+1} = \left\{\left[1-\delta+AN_t^{1-\alpha}\exp\left(z_{1,t}\right)\right]K_t - C_t\right\}\exp\left(z_{2,t+1}\right). \tag{31}$$

The various variables of this model economy evolve according to the nonlinear system of Eqs. (29), (30), (31), (23) and (24). It can be shown from these five equations that the model economy exhibits endogenous balanced growth such that labor input (N_t) is stationary around a constant mean, but output, consumption, investment and capital all grow at an average rate γ. That is,

$$J_t^* = (1+\gamma)\, J_{t-1}^*, \qquad J = Y, C, I, K, \tag{32}$$

where J_t^* $(J = Y, C, I, K)$ denotes the value of variable J_t at the balanced growth path, with $z_{1,t} = z_{2,t} = 0$, and γ (the endogenously determined growth rate) is given by

$$\gamma = \beta\left[1-\delta+\alpha A^{\frac{1+\omega}{\alpha+\omega}}\left(\frac{1-\alpha}{1+\omega}\right)^{\frac{1-\alpha}{\alpha+\omega}}\right] - 1. \tag{33}$$

To guarantee a meaningful growth model, it is assumed that the various parameters are such that the right-hand side of (33) is positive.

The next step is to loglinearize the dynamic system around the balanced growth path. As observed in Appendix B, this step is extremely tedious.[12] After using (B6) to express the deviation of labor supply in terms of the productivity shock, the

[12]Even though the model in Hercowitz and Sampson (1991) exhibits endogenous growth (and therefore, according to Lau (1999), the coefficient of lagged capital in the dynamic equation of

approximate loglinear system (in terms of capital, consumption and the exogenous shock processes) can be reduced to the following two expectational equations:

$$(\ln C_t - \ln C_t^*) - \theta_{CK} (\ln K_t - \ln K_t^*) - \theta_{C0} z_{1,t}$$

$$= E_t \left[\left(\ln C_{t+1} - \ln C_{t+1}^* \right) - \theta_{CK} \left(\ln K_{t+1} - \ln K_{t+1}^* \right) - \theta_{C1} z_{1,t+1} - \theta_{C2} z_{2,t+1} \right], \quad (34)$$

and

$$\ln K_{t+1} - \ln K_{t+1}^* = \theta_{KK} (\ln K_t - \ln K_t^*) - \theta_{KC} (\ln C_t - \ln C_t^*) + \theta_{K1} z_{1,t} + z_{2,t+1}, \quad (35)$$

where the coefficients in (34) and (35) are given in Appendix B.

Finally, solving the approximate loglinear system of equations, (34) and (35) with (23) and (24), gives the equation of motion of the capital stock as

$$\ln K_{t+1} = \gamma + \ln K_t + \frac{\theta_{K1} - \theta_{KC}\theta_{C0} + \phi_1 (\theta_{KC}\theta_{C1} - \theta_{K1})}{1 + \theta_{KC} - \theta_{KC}\theta_{CK} - \phi_1} z_{1,t} + z_{2,t+1}. \quad (36)$$

5.2. The alternative method

As in Section 3, the first step of the alternative method is to loglinearize the capital accumulation equation around the balanced growth path. (Since the utility function (19) is logarithmic, there is no further need to loglinearize it.) It can be shown that along the balanced growth path, investment and capital are related by:

$$(\delta + \gamma) K_t^* = I_t^*. \quad (21a)$$

capital is exactly 1), the loglinearization of the dynamic system around the balanced growth path is still tremendously tedious. One is tempted to give up the conventional analytical approach and use computational methods to obtain the solution. With great perseverance, I press on the analysis so as to make clear that while the conventional approximate method works in principle, the expressions become very convoluted when more elements (such as labor-leisure choice and stochastic shocks) are added to the model. See also footnote 10.

Loglinearizing the capital accumulation equation (21) around the balanced growth path leads to

$$\ln K_{t+1} = \ln(1+\gamma) - \delta_n \ln(\delta+\gamma) + (1-\delta_n)\ln K_t + \delta_n \ln I_t + z_{2,t+1}, \tag{21b}$$

where δ_n, which depends on the endogenously determined growth rate γ and is similar in form to (12), is defined as:

$$\delta_n = \frac{\delta+\gamma}{1+\gamma}. \tag{37}$$

Note that Eq. (2) of Hercowitz and Sampson (1991) is essentially the same as Eq. (21b) of this paper, provided that appropriate restrictions are imposed on the coefficients. Therefore, instead of simply assuming a nonstandard capital accumulation equation, an alternative interpretation of Eq. (2) of Hercowitz and Sampson (1991) is the loglinear approximation of (a stochastic version of) the standard capital accumulation equation around the balanced growth path.[13]

Using the method suggested in Section 3, the approximate solution is obtained by maximizing (25) subject to (23), (24) and (21b), where (21b) is the loglinear approximation of the capital accumulation equation (21). It is shown in Appendix C that the optimal policy rules are given by (27a) and

$$I_t = \left[\frac{\beta\alpha\delta_n}{1-\beta(1-\delta_n)}\right] Y_t. \tag{38}$$

Therefore, according to the alternative approximate solution, the endogenously determined saving rates of the Hercowitz and Sampson (1991) model are time-invariant, which is given by the bracket term in (38).

Substituting (38) into (21b) gives the equilibrium dynamic equation of the capital stock as:

$$\ln K_{t+1} = \gamma + \ln K_t + \frac{\delta_n(1+\omega)}{\alpha+\omega} z_{1,t} + z_{2,t+1}, \tag{39}$$

[13] A similar interpretation applies to Eq. (3) of Kocherlakota and Yi (1997) and Eq. (3) of Collard (1999).

where γ is given by (33).

Eqs. (38) and (39) summarize the main features of the alternative approximate solution of this economy. The derivation of other equations is similar to that in Hercowitz and Sampson (1991) and will not be presented.

5.3. A comparison of the two methods

The first comparison between the conventional and alternative methods is about the equilibrium equation of motion. Comparing (36) and (39) suggests that for the Hercowitz and Sampson (1991) model with endogenous growth, the coefficient of lagged capital (the state variable) is the same for both methods. Thus, the long-run growth properties of the Hercowitz and Sampson (1991) model are not affected by the choice of the approximate solution procedures. On the other hand, while the form of the dependence of $\ln K$ on z_1 is the same (both depending on the *current* productivity shock only), the precise coefficients are different for the two methods. Fortunately, this difference is probably of minor concerns, especially for empirical work. What matters most for empirical analysis is the choice of the appropriate functional form for estimation; the coefficients are estimated from the data anyway. The two methods suggest the same form of estimation equation for, say, the effect of productivity shock on capital stock (or output).

The second, and more important, comparison is about the method. When applying to a dynamic stochastic general equilibrium model with a number of features (such as the Hercowitz and Sampson (1991) model with labor-leisure choice and stochastic shocks), the conventional method leads to very convoluted expressions; see especially (B7) and (B8). On the other hand, a major advantage of the alternative method is its simplicity, which arises from the time-invariant (endogenous) saving rates. Such simplicity, which facilitates further derivation of analytical results, represents a major advantage of this alternative solution procedure.

6. CONCLUDING REMARKS

Campbell (1994) argues that it is beneficial to use an analytical approach to examine dynamic stochastic general equilibrium models and suggests loglinearizing the Euler equation and the capital accumulation equation. An approximate analytical solution not only enables the reader to trace the dynamic effects of the underlying economic shocks but also is useful to welfare analysis, such as examining issues related to the maximized utility level of the representative agent.

This paper further suggests a simpler approximate solution procedure to the class of dynamic stochastic general equilibrium models by integrating the idea presented in Long and Plosser (1983) and Campbell (1994). The suggested approximate analytical solution is obtained by loglinearizing the utility function and the capital accumulation equation around the balanced growth path, and it has the property that the endogenously determined saving rates are time-invariant. Compared with the conventional approximate analytical solution (which quickly becomes complicated when the model contains more features, such as labor-leisure choice and stochastic shocks), the advantage of the alternative approximate solution procedure is its simplicity. As a result, it is relatively easy to obtain further analytical results based on this solution procedure and to provide insight about the economic issues under investigation.

The advantages of a simple analytical solution have to be compared with the advantages of other possible methods, in order to fully assess its contributions. Based on a comparison of the solution to the Ramsey-Cass-Koopmans model suggested in Campbell (1994) and that suggested in this paper, together with the time-series properties of stochastic endogenous growth models (such as those presented in Lau, 1999), it is suggested that the solution procedure proposed in this paper is most useful for the class of dynamic stochastic general equilibrium models exhibiting endogenous growth.[14] Applying the alternative solution procedure to the model examined in Hercowitz and Sampson (1991) demonstrates this point.

A by-product of the analysis of this paper is the reconciliation of the approach that uses a Cobb-Douglas capital accumulation equation (as in Hercowitz and Sampson, 1991) with the standard approach. As long as the Cobb-Douglas capital accumulation equation (with appropriate restrictions on the coefficients) is interpreted as the loglinear approximation of the standard capital accumulation equation around the balanced growth path, the main results of those papers using the nonstandard approach (for example, Hercowitz and Sampson, 1991; Kocherlakota and Yi, 1997; Collard, 1999) will still hold (approximately) if the standard approach is used.[15]

APPENDIX A: SECTION 3

The solution can be obtained by the method of undetermined coefficients. Guess that the value function is of the following form:

$$V(k_t) = a + b \ln k_t, \tag{A1}$$

where a and b are unknown coefficients to be determined. Under this conjecture, the right-hand side of (16) is given by:

$$c^{\left(1-\frac{1}{\sigma}\right)} \ln\left(Ak_t^{\alpha} - i_t\right) + \beta_x\{a + b\left[(1-\delta_x)\ln k_t + \delta_x \ln i_t - \delta_x \ln\left(\delta + \gamma_x\right)\right]\}. \tag{A2}$$

[14]Equivalently, the approximate solution procedure suggested in this paper is most useful for those dynamic stochastic general equilibrium models in which the endogenously determined saving rates are 'not too time-varying'. Stated in this way, the above point is similar in spirit to the point mentioned, in the context of the intertemporal asset pricing model, by Campbell (1993, p. 502): "... a log-linear approximation to the intertemporal budget constraint ... will be accurate when the log consumption-wealth ratio is 'not too' variable."

[15]In all these papers, a log utility function is used. The analysis of this paper further suggests that their results can easily be extended to the class of CIES utility functions.

Differentiating (A2) with respect to i_t gives the optimal policy rule as:

$$i_t = \left[\frac{\beta_x \delta_x b}{c^{\left(1-\frac{1}{\sigma}\right)} + \beta_x \delta_x b} \right] A k_t^{\alpha}. \tag{A3}$$

The unknown coefficients in (A1) are then determined by substituting (5b), (A1) and (A3) into (16). In particular, it is found that $b = \left[\frac{\alpha}{1-\beta_x[1-(1-\alpha)\delta_x]} \right] c^{\left(1-\frac{1}{\sigma}\right)}$. Therefore, the optimal policy rule is expressed in terms of the underlying parameters as (17).

APPENDIX B: SECTION 5.1

By focusing on consumption (instead of investment) as the choice variable and by denoting

$$U\left(C_t, N_t; \bar{K}_t\right) = \ln\left(C_t - \bar{K}_t N_t^{1+\omega}\right),$$

the Bellman equation of optimality can be rewritten as:

$$V\left(K_t, \bar{K}_t, z_{1,t}, z_{2,t}\right) = \max_{C_t, N_t} \left[U\left(C_t, N_t; \bar{K}_t\right) + \beta E_t V\left(K_{t+1}, \bar{K}_{t+1}, z_{1,t+1}, z_{2,t+1}\right) \right],$$

subject to (23), (24),

$$K_{t+1} = \left[(1-\delta) K_t + Y_t - C_t\right] \exp\left(z_{2,t+1}\right),$$

and

$$Y_t = A K_t^{\alpha} \left(\bar{K}_t N_t\right)^{1-\alpha} \exp\left(z_{1,t}\right),$$

which is formed by substituting (22) into (20).

Following standard procedures, it can be shown that the first-order conditions are:

$$\frac{\partial U\left(C_t, N_t; \bar{K}_t\right)}{\partial C_t} = \beta E_t \left[\left(1 - \delta + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\right) \frac{\partial U\left(C_{t+1}, N_{t+1}; \bar{K}_{t+1}\right)}{\partial C_{t+1}} \exp\left(z_{2,t+1}\right) \right], \tag{B1}$$

and

$$-\frac{\partial U\left(C_t, N_t; \bar{K}_t\right)}{\partial N_t} = \frac{\partial Y_t}{\partial N_t} \frac{\partial U\left(C_t, N_t; \bar{K}_t\right)}{\partial C_t}. \tag{B2}$$

Substituting the functional forms for utility and production functions, (B1) and (B2) become (26) and (27) respectively.

Along the balanced growth path, (29), (30) and (31) become

$$N_t^* = \left(\frac{1-\alpha}{1+\omega}A\right)^{\frac{1}{\alpha+\omega}} \equiv N^*, \tag{B3}$$

$$\frac{1}{C_t^* - K_t^* (N^*)^{1+\omega}} = \beta \frac{1-\delta+\alpha A (N^*)^{1-\alpha}}{C_{t+1}^* - K_{t+1}^* (N^*)^{1+\omega}}, \tag{B4}$$

and

$$K_{t+1}^* = \left\{ \left[1-\delta + A (N^*)^{1-\alpha}\right] K_t^* - C_t^* \right\}, \tag{B5}$$

where the variables J_t^* $(J = Y, C, I, K)$ obey (32) and (33). Combining (32), (B3) and (B4) lead to (33). Using (32) and (B5) gives

$$C_t^* = \left[A (N^*)^{1-\alpha} - \delta - \gamma\right] K_t^*. \tag{B5a}$$

The next step is to loglinearize the dynamic system around the balanced growth path. Loglinearizing (29) gives

$$\ln N_t - \ln N^* = \frac{1}{\alpha+\omega} z_{1,t}. \tag{B6}$$

Loglinearizing (30) and simplifying gives

$$\begin{aligned}
&\left[A (N^*)^{1-\alpha} - \delta - \gamma\right] (\ln C_t - \ln C_t^*) - (N^*)^{1+\omega} (\ln K_t - \ln K_t^*) - (1+\omega) (N^*)^{1+\omega} (\ln N_t - \ln N^*) \\
&= E_t \Big\{ \left[A (N^*)^{1-\alpha} - \delta - \gamma\right] \left(\ln C_{t+1} - \ln C_{t+1}^*\right) - (N^*)^{1+\omega} \left(\ln K_{t+1} - \ln K_{t+1}^*\right) \\
&\quad - (1+\omega) (N^*)^{1+\omega} (\ln N_{t+1} - \ln N^*) - \left[A (N^*)^{1-\alpha} - (N^*)^{1+\omega} - \delta - \gamma\right] z_{2,t+1} \\
&- \left[\frac{1+\gamma-\beta(1-\delta)}{1+\gamma}\right] \left[A (N^*)^{1-\alpha} - (N^*)^{1+\omega} - \delta - \gamma\right] \left[(1-\alpha)(\ln N_{t+1} - \ln N^*) + z_{1,t+1}\right] \Big\}
\end{aligned} \tag{B7}$$

Loglinearizing (31) and simplifying gives

$$\left(\ln K_{t+1} - \ln K_{t+1}^*\right) = z_{2,t+1} + \left[1 - \delta_n + \frac{1-\beta(1-\delta_n)}{\beta\alpha}\right] (\ln K_t - \ln K_t^*)$$

$$+\left[\frac{1-\beta\left(1-\delta_n\right)}{\beta\alpha}\right]\left[\left(1-\alpha\right)\left(\ln N_t-\ln N^*\right)+z_{1,t}\right]-\left[\frac{1-\beta\left(1-\delta_n\right)}{\beta\alpha}-\delta_n\right]\left(\ln C_t-\ln C_t^*\right). \tag{B8}$$

Substituting (B6) into (B7) gives (34) and substituting (B6) into (B8) gives (35), where the coefficients of (34) and (35) are given by:

$$\theta_{CK}=\frac{\left(N^*\right)^{1+\omega}}{A\left(N^*\right)^{1-\alpha}-\delta-\gamma},$$

$$\theta_{C0}=\left(\frac{1+\omega}{\alpha+\omega}\right)\theta_{CK},$$

$$\theta_{C1}=\left(\frac{1+\omega}{\alpha+\omega}\right)\left\{\left[1-\beta\left(1-\delta_n\right)\right]+\beta\left(1-\delta_n\right)\theta_{CK}\right\},$$

$$\theta_{C2}=1-\theta_{CK},$$

$$\theta_{KK}=1-\delta_n+\frac{1-\beta\left(1-\delta_n\right)}{\beta\alpha},$$

$$\theta_{KC}=\theta_{KK}-1,$$

$$\theta_{K1}=\left[\frac{1-\beta\left(1-\delta_n\right)}{\beta\alpha}\right]\left(\frac{1+\omega}{\alpha+\omega}\right),$$

and δ_n is given by (37).

As in Section 2, one can use the method of undetermined coefficients to solve the approximate loglinear system of equations: (23), (24), (34) and (35). Conjecture that

$$\ln C_t-\ln C_t^*=\eta_{CK}\left(\ln K_t-\ln K_t^*\right)+\eta_{C1}z_{1,t}+\eta_{C2}z_{2,t}, \tag{B9}$$

and

$$\ln K_{t+1}-\ln K_{t+1}^*=\eta_{KK}\left(\ln K_t-\ln K_t^*\right)+\eta_{K1}z_{1,t}+\eta_{K2}z_{2,t}+\eta_{K3}z_{2,t+1}. \tag{B10}$$

Substituting (B9) and (B10) into (34) and (35) and equating coefficients, it can be shown that

$$\eta_{KK}=\eta_{K3}=\eta_{CK}=1,$$

$$\eta_{K2}=\eta_{C2}=0,$$

$$\eta_{K1} = \frac{\theta_{K1} - \theta_{KC}\theta_{C0} + \phi_1 \left(\theta_{KC}\theta_{C1} - \theta_{K1}\right)}{1 + \theta_{KC} - \theta_{KC}\theta_{CK} - \phi_1},$$

and

$$\eta_{C1} = \frac{\theta_{K1} - \eta_{K1}}{\theta_{KC}}.$$

Substituting these parameters into (B10) and using (32) gives (36).

APPENDIX C: SECTION 5.2

Guess that the value function is of the following form:

$$V\left(K_t, \bar{K}_t, z_{1,t}, z_{2,t}\right) = d_0 + d_1 \ln K_t + d_2 \ln \bar{K}_t + f_1 z_{1,t} + f_2 z_{2,t}, \tag{C1}$$

where d_0, d_1, d_2, f_1 and f_2 are unknown coefficients to be determined. Under this conjecture, the right-hand side of (25) is given by:

$$\ln\left[AK_t^{\alpha}\left(\bar{K}_t N_t\right)^{1-\alpha}\exp\left(z_{1,t}\right) - I_t - \bar{K}_t N_t^{1+\omega}\right]$$
$$+\beta E_t\left\{d_0 + d_1\left[\ln\left(1+\gamma\right) - \delta_n \ln\left(\delta+\gamma\right) + \left(1-\delta_n\right)\ln K_t + \delta_n \ln I_t + z_{2,t+1}\right]\right.$$
$$\left. + d_2 \ln \bar{K}_{t+1} + f_1 z_{1,t+1} + f_2 z_{2,t+1}\right\}. \tag{C2}$$

Differentiating (C2) with respect to N_t gives the optimal choice of N_t as (27a), and differentiating (C2) with respect to I_t gives

$$I_t = \left(\frac{\beta\delta_n d_1}{1+\beta\delta_n d_1}\right)\left(Y_t - \bar{K}_t N_t^{1+\omega}\right). \tag{C3}$$

It can be shown from (20), (22), (27a) and (28) that

$$Y_t - \bar{K}_t N_t^{1+\omega} = A^{\frac{1+\omega}{\alpha+\omega}}\left[\left(\frac{1-\alpha}{1+\omega}\right)^{\frac{1-\alpha}{\alpha+\omega}} - \left(\frac{1-\alpha}{1+\omega}\right)^{\frac{1+\omega}{\alpha+\omega}}\right]\left(K_t\right)^{\frac{\alpha(1+\omega)}{\alpha+\omega}}\left(\bar{K}_t\right)^{\frac{\omega(1-\alpha)}{\alpha+\omega}}\left(\exp z_{1,t}\right)^{\frac{1+\omega}{\alpha+\omega}}$$
$$= \left(\frac{\alpha+\omega}{1+\omega}\right)Y_t. \tag{C4}$$

Substituting (C4) into (C3) gives

$$I_t = \left(\frac{\beta\delta_n d_1}{1+\beta\delta_n d_1}\right)\left(\frac{\alpha+\omega}{1+\omega}\right)Y_t.$$

Substituting the relevant equations into (25), the unknown coefficients in (C1) can be determined. In particular, it is found that $d_1 = \frac{\alpha(1+\omega)}{(\alpha+\omega)-\beta[\alpha+\omega(1-\delta_n+\alpha\delta_n)]}$. As a result, investment is given by (38).

Finally, substituting (38) into (21b) gives (39), since it can be shown that

$$\ln(1+\gamma) - \delta_n \ln(\delta+\gamma) + \delta_n \left[\ln\left(\frac{\beta\alpha\delta_n}{1-\beta(1-\delta_n)}\right) + \frac{1+\omega}{\alpha+\omega}\ln A + \frac{1-\alpha}{\alpha+\omega}\ln\left(\frac{1-\alpha}{1+\omega}\right)\right] = \gamma.$$

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Table 1

The elasticity of capital with respect to its lag term in the two approximate solutions

α	σ			
	0.5	1	2	5
0.33	0.971 0.980	0.957 0.980	0.938 0.980	0.902 0.980
0.42	0.978 0.983	0.967 0.983	0.952 0.983	0.922 0.983
0.67	0.990 0.990	0.985 0.990	0.977 0.990	0.962 0.990
0.80	0.995 0.994	0.992 0.994	0.987 0.994	0.977 0.994
0.95	0.999 0.999	0.998 0.999	0.996 0.999	0.993 0.999

The top number in each cell is η_{kk} according to (14) and the bottom number is $1-(1-\alpha)\delta_x$ according to (18). Parameter α is the exponent on capital in the production function, and σ is the intertemporal elasticity of substitution. The assumed value of other parameters are $\delta = 0.025$ and $\gamma_x = 0.005$. Moreover, in order to make the steady-state real interest rate equal to 0.015, the implied value of β_x is 0.990.