Cross-Hedging with Currency Forward Contracts *

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Abstract

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JEL classification: D81; F23; F31

Keywords: Cross-hedging; Regression dependence; Relative risk aversion

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1. Introduction

In many less developed countries (LDCs) wherein capital markets are embryonic and foreign exchange markets are heavily controlled, currency derivative markets for forwards, futures, and options are seldom readily available. Even if some LDCs have currency forward contracts, these contracts are deemed to be forward-cover insurance schemes that are not governed by market forces (see Jacque, 1996). Also, in many of the newly industrializing countries of Latin America and Asia Pacific, currency derivative markets are just starting to develop in a rather slow pace (see Eiteman, Stonehill, and Moffett, 2004). International firms that expose to currencies of these countries thus have to avail themselves of derivative securities on related currencies to hedge against their exchange rate risk exposure. Such an exchange rate risk management technique is referred to as "cross-hedging."

The purpose of this paper is to offer analytical insights into the optimal cross-hedging strategies of international firms. To this end, we develop an expected utility model of an international firm facing exchange rate risk exposure to a foreign currency cash flow. Financial markets are incomplete in that there are no hedging instruments directly related to the home currency. There is, however, an unbiased currency forward market between the foreign currency and a third currency accessible to the firm. A triangular parity condition is assumed to hold among the home, foreign, and third currencies, thereby making cross-hedging provided by the available, yet incomplete, currency forward market useful to the firm in reducing its exchange rate risk exposure.

We show that the firm optimally opts for a full-hedge if the spot exchange rate of the home currency against the third currency and that of the third currency against the foreign currency are independent or if the firm's utility function is logarithmic. This is analogous to the celebrated full-hedging theorem (see, e.g., Katz and Paroush, 1979; Kawai and Zilcha, 1986; and Broll and Eckwert, 1996). If the two spot exchange rates are positively (negatively) correlated in the sense of regression dependence (see Lehmann, 1966), we show that the firm's optimal forward position is an under-hedge or an over-hedge, depending on whether the firm's Arrow-Pratt measure of relative risk aversion is everywhere less (greater) or greater (less) than unity, respectively. Regression dependence describes the form of non-linear dependence structure of two random variables in general and how they behave together when their realizations are simultaneously small or large in particular. Shea (1979) shows that positive (negative) regression dependence of two random variables implies positive (negative) correlation between them, albeit not the converse.

Since the seminal paper of Anderson and Danthine (1981), cross-hedging has become an important strand of the hedging literature. Broll and Eckwert (1996), Adam-Müller (1997), and Broll, Wong, and Zilcha (1999) look at the case where the multiple sources of uncertainty are independent. Broll, Wahl, and Zilcha (1995), Broll and Wong (1999), Chang and Wong (2003), and Wong (2003) allow linear dependence structure to accommodate correlated sources of uncertainty. The contribution of this paper is to introduce general non-linear dependence structure that enables intuitive prescriptions of cross-hedging strategies for international firms (see also Wong, 2006).

The rest of the paper is organized as follows. In the next section, we develop an expected utility model of an international firm facing exchange rate risk exposure to a foreign currency cash flow and cross-hedging opportunities. Section 3 derives the firm's optimal cross-hedge position when an unbiased currency forward market between the foreign currency and a third currency is available. The final section concludes.

2. The model

Consider an international firm with an operation domiciled in a foreign country. To begin, the firm looks forward to receiving a net cash flow, x, from its foreign operation, where x is denominated in the foreign currency. This foreign currency cash flow, x, is fixed and known to the firm *ex ante*.

There is a third country that has a currency forward market to which the firm has access for cross-hedging purposes. We index the home country by 0, the third country by 1, and the foreign country by 2. Exchange rate uncertainty is modeled by three positive random variables, $\tilde{e}_{01}$, $\tilde{e}_{02}$, and $\tilde{e}_{12}$, where $\tilde{e}_{ij}$ is the spot exchange rate expressed in units of country i's currency per unit of country j's currency. Based on $\tilde{e}_{01}$ and $\tilde{e}_{12}$, we can define a cross rate of the home currency against the foreign currency as $\tilde{e}_{01}\tilde{e}_{12}$. It follows immediately from the law of one price that $\tilde{e}_{02} = \tilde{e}_{01}\tilde{e}_{12}$. Throughout the paper, random variables have a tilde ($\sim$) while their realizations do not.

We assume that there are no hedging instruments directly related to the home currency. There is, however, a currency forward market between the foreign and third currencies. To cross-hedge against its exchange rate risk exposure, the firm sells (purchases if negative) z units of the third currency forward at the pre-determined forward exchange rate, e_{12}^f. The firm's random profit, denominated in the home currency, is therefore given by

$$\tilde{\pi} = \tilde{e}_{01}\tilde{e}_{12}x + \tilde{e}_{01}(e_{12}^f - \tilde{e}_{12})z, \qquad (1)$$

where $\tilde{e}_{01}(e_{12}^f - \tilde{e}_{12})z$ is the gain or loss due to the forward position, z.

The firm possesses a von Neumann-Morgenstern utility function, $u(\pi)$, defined over its home currency profit, π, with $u'(\pi) > 0$ and $u''(\pi) < 0$, indicating the presence of risk aversion. The firm's *ex ante* decision problem is to choose its forward position, z, so as to maximize the expected utility of its home currency profit defined in equation (1):

$$\max_z \int_{\underline{e}_{01}}^{\overline{e}_{01}} \int_{\underline{e}_{12}}^{\overline{e}_{12}} u[e_{01}e_{12}x + e_{01}(e_{12}^f - e_{12})z] \, \mathrm{d}F(e_{01}|e_{12}) \, \mathrm{d}G(e_{12}). \tag{2}$$

where $F(e_{01}|e_{12}) : [\underline{e}_{01}, \overline{e}_{01}] \to [0, 1]$ is the cumulative distribution function (CDF) of $\tilde{e}_{01}$ given a realized value of $\tilde{e}_{12}$, and $G(e_{12}) : [\underline{e}_{12}, \overline{e}_{12}] \to [0, 1]$ is the marginal CDF of $\tilde{e}_{12}$.

3. Optimal cross-hedging decisions

The first-order condition for program (2) is given by

$$\int_{\underline{e}_{01}}^{\overline{e}_{01}} \int_{\underline{e}_{12}}^{\overline{e}_{12}} u'[e_{01}e_{12}x + e_{01}(e_{12}^f - e_{12})z^*]e_{01}(e_{12}^f - e_{12}) \, \mathrm{d}F(e_{01}|e_{12}) \, \mathrm{d}G(e_{12}) = 0, \tag{3}$$

where z^* is the firm's optimal forward position. The second-order condition for program (2) is satisfied given risk aversion.

The firm's optimal forward position, z^*, is implicitly defined in equation (3). Among other things, z^* would depend on whether the firm perceives the forward exchange rate, e_{12}^f, as unbiased or biased. In the former unbiased case, z^* should reflect solely the hedging motive of the firm. In the latter biased case, z^* reflects also the speculative motive of the firm. Since our objective is to show the cross-hedging role of the currency forward contracts, we shall hereafter assume that e_{12}^f is perceived as unbiased by the firm. That is, e_{12}^f is set equal to the unconditional expected value of $\tilde{e}_{12}$.

Given the unbiasedness of the currency forward contracts, we can write equation (3) as

$$\int_{\underline{e}_{12}}^{\overline{e}_{12}} [N(e_{12}) - N(e_{12}^f)](e_{12}^f - e_{12}) \, \mathrm{d}G(e_{12}) = 0, \tag{4}$$

where $N(e_{12})$ is the conditional expectation of $u'[\tilde{e}_{01}e_{12}x + \tilde{e}_{01}(e^f_{12} - e_{12})z^*]\tilde{e}_{01}$ with respect to $F(e_{01}|e_{12})$:

$$N(e_{12}) = \int_{\underline{e}_{01}}^{\overline{e}_{01}} u'[e_{01}e_{12}x + e_{01}(e^f_{12} - e_{12})z^*]e_{01}\ \mathrm{d}F(e_{01}|e_{12}). \tag{5}$$

Using integration by parts, we can write equation (5) as

$$N(e_{12}) = u'[\overline{e}_{01}e_{12}x + \overline{e}_{01}(e^f_{12} - e_{12})z^*]\overline{e}_{01} - \int_{\underline{e}_{01}}^{\overline{e}_{01}} u'[e_{01}e_{12}x + e_{01}(e^f_{12} - e_{12})z^*]$$

$$\times\{1 - r[e_{01}e_{12}x + e_{01}(e^f_{12} - e_{12})z^*]\}F(e_{01}|e_{12})\ \mathrm{d}e_{01}, \tag{6}$$

where $r(\pi) = -\pi u''(\pi)/u'(\pi)$ is the firm's Arrow-Pratt measure of relative risk aversion.

If $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are independent, we have $F(e_{01}|e_{12}) = F(e_{01}|e^f_{12})$ for all $(e_{01}, e_{12}) \in [\underline{e}_{01}, \overline{e}_{01}] \times [\underline{e}_{12}, \overline{e}_{12}]$. If $r(\pi) \equiv 1$ for all π, the second term on the right-hand side of equation (6) vanishes. In either case, it follows from equation (6) that $N(e_{12}) = N(e^f_{12})$ for all $e_{12} \in [\underline{e}_{12}, \overline{e}_{12}]$ when $z^* = x$. Thus, $z^* = x$ is the solution to equation (4), thereby invoking our first proposition.

Proposition 1. *Given that the international firm is allowed to cross-hedge via trading the unbiased currency forward contracts for $\tilde{e}_{12}$. If $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are independent or if the firm's Arrow-Pratt measure of relative risk aversion is everywhere equal to unity, then the firm optimally opts for a full-hedge, i.e., $z^* = x$.*

The intuition of Proposition 1 is as follows. When $z = x$, the firm's home currency profit becomes $\tilde{e}_{01}e^f_{12}x$. If $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are independent, there would be no residual risk that is hedgeable by the currency forward contracts, thereby rendering the optimality of a full-hedge. On the other hand, if the firm's Arrow-Pratt measure of relative risk aversion is everywhere equal to unity, the firm's utility function must be logarithmic. In this case, the firm's *ex ante* decision problem is given by

$$\max_z\ \ln[\tilde{e}_{01}\tilde{e}_{12}x + \tilde{e}_{01}(e^f_{12} - \tilde{e}_{12})z] = \ln(\tilde{e}_{01}) + \ln[\tilde{e}_{12}x + (e^f_{12} - \tilde{e}_{12})z],$$

which reduces to the usual decision problem under a single source of uncertainty. Thus, the full-hedging theorem applies, implying that $z^* = x$ is indeed optimal.

To characterize the firm's optimal forward position when $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are not independent, we need to impose some dependence structure on these two random spot exchange rates. The following concept of bivariate dependence is taken from Lehmann (1966).

Definition 1. We say that $\tilde{e}_{01}$ *is positively (negatively) regression dependent on* $\tilde{e}_{12}$ *if, and only if, the conditional CDF of* $\tilde{e}_{01}$ *improves (deteriorates) in the sense of first-degree stochastic dominance as the realized value of* $\tilde{e}_{12}$ *increases; i.e.,* $F_{e_{12}}(e_{01}|e_{12}) \leq (\geq)\ 0$ *for all* $(e_{01}, e_{12}) \in [\underline{e}_{01}, \overline{e}_{01}] \times [\underline{e}_{12}, \overline{e}_{12}]$, *where the subscript,* e_{12}, *denotes a partial derivative.*

The positive (negative) regression dependence has the intuition that $\tilde{e}_{01}$ and $\tilde{e}_{12}$ tend to move in the same direction (opposite directions). Using Chebyshev inequality, Shea (1979) shows that Definition 1 implies that $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are positively (negatively) correlated (the converse is not true though). For example, if $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are normally distributed, they will be positively or negatively regression dependent according to whether the correlation coefficient is positive or negative, respectively. Aboudi and Thon (1995) argue that the concept of regression dependence is the most suitable concept of bivariate dependence for applications to the theory of choice under multiple sources of uncertainty (see also Wong, 1996, 2006).

Proposition 2. *Given that the international firm is allowed to cross-hedge via trading the unbiased currency forward contracts for* $\tilde{e}_{12}$. *If* $\tilde{e}_{01}$ *is positively (negatively) regression dependent on* $\tilde{e}_{12}$, *then the firm optimally opts for an under-hedge or an over-hedge depending on whether the firm's Arrow-Pratt measure of relative risk aversion is everywhere less (greater) than or greater (less) than unity, respectively.*

Proof. Partially differentiating equation (6) with respect to e_{12} yields

$$N'(e_{12}) = \int_{\underline{e}_{01}}^{\overline{e}_{01}} u''[e_{01}e_{12}x + e_{01}(e_{12}^f - e_{12})z^*]e_{01}^2(x - z^*)\ \mathrm{d}F(e_{01}|e_{12})$$

$$- \int_{\underline{e}_{01}}^{\overline{e}_{01}} u'[e_{01}e_{12}x + e_{01}(e_{12}^f - e_{12})z^*]$$

$$\times\{1 - r[e_{01}e_{12}x + e_{01}(e_{12}^f - e_{12})z^*]\}F_{e_{12}}(e_{01}|e_{12})\ \mathrm{d}e_{01}. \quad (7)$$

Consider first the case that $\tilde{e}_{01}$ is positively regression dependent on $\tilde{e}_{12}$ and $r(\pi) > (<)\ 1$ for all π. Suppose that $z^* \leq (\geq)\ x$. Then, equation (7) implies that $N'(e_{12}) \leq (\geq)\ 0$. The left-hand side of equation (4) becomes positive (negative), a contradiction. Hence, $z^* > (<)\ x$.

Now consider the case that $\tilde{e}_{01}$ is negatively regression dependent on $\tilde{e}_{12}$ and $r(\pi) > (<)\ 1$ for all π. Suppose that $z^* \geq (\leq)\ x$. Then, equation (7) implies that $N'(e_{12}) \geq (\leq)\ 0$. The left-hand side of equation (4) becomes negative (positive), a contradiction. Hence, $z^* < (>)\ x$. □

To see the intuition of Proposition 2, we use the covariance operator, $\mathrm{Cov}(\cdot,\cdot)$, to recast equation (3) as

$$\mathrm{Cov}\{u'[\tilde{e}_{01}\tilde{e}_{12}x + \tilde{e}_{01}(e_{12}^f - \tilde{e}_{12})z^*]\tilde{e}_{01}, \tilde{e}_{12}\} = 0. \quad (8)$$

Since covariances can be interpreted as marginal variances, equation (8) implies that the optimal forward position , z^*, minimizes the variance of the product of the marginal utility and $\tilde{e}_{01}$ across different realizations of $\tilde{e}_{12}$. Consider first the case that $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are positively regression dependent. At $z^* = x$, the left-hand side of equation (8) becomes $\mathrm{Cov}[u'(\tilde{e}_{01}e_{12}^f x)\tilde{e}_{01}, \tilde{e}_{12}]$, which can be positive or negative since $\mathrm{Cov}[u'(\tilde{e}_{01}e_{12}^f x), \tilde{e}_{12}] < 0$ but $\mathrm{Cov}(\tilde{e}_{01}, \tilde{e}_{12}) > 0$. The first effect dominates (is dominated by) the second effect if the firm is sufficiently (not too) risk averse, i.e., the firm's Arrow-Pratt measure of relative risk aversion is everywhere greater (less) than unity. Thus, we have $\mathrm{Cov}[u'(\tilde{e}_{01}e_{12}^f x)\tilde{e}_{01}, \tilde{e}_{12}] < (>)\ 0$, thereby rendering the optimality of an over-hedge (under-hedge), i.e., $z^* > (<) x$. The results in the case that $\tilde{e}_{01}$ and $\tilde{e}_{12}$ are negatively regression dependent can be understood analogously.

4. Conclusions

In this paper, we have examined the hedging decision of an international firm facing exchange rate risk exposure to a foreign currency cash flow. Financial markets are incomplete in that the firm has access to an unbiased currency forward market between the foreign currency and a third currency only. A triangular parity condition holds among the home, foreign, and third currencies, thereby making cross-hedging provided by the available, yet incomplete, currency forward market useful to the firm in reducing its exchange rate risk exposure. If the spot exchange rate of the home currency against the third currency and that of the third currency against the foreign currency are positively (negatively) correlated in the sense of regression dependence (see Lehmann, 1966), we have shown that the firm optimally opts for an under-hedge or an over-hedge for cross-hedging purposes depending on whether the firm's Arrow-Pratt measure of relative risk aversion is everywhere less (greater) or greater (less) than unity, respectively.

References

Aboudi, R. and D. Thon (1995) Second-Degree Stochastic Dominance Decisions and Random Initial Wealth with Applications to the Economics of Insurance, *Journal of Risk and Insurance*, **62**, 30–49.

Adam-Müller, A.F.A. (1997) Export and Hedging Decisions under Revenue and Exchange Rate Risk: A Note, *European Economic Review*, **41**, 1421–1426.

Anderson, R.W. and J.-P. Danthine (1981) Cross Hedging, *Journal of Political Economy*, **89**, 1182–1196.

Broll, U. and B. Eckwert (1996) Cross-Hedging of Exchange-Rate Risk, *Review of International Economics*, **4**, 282–286.

Broll, U., J.E. Wahl and I. Zilcha (1995) Indirect Hedging of Exchange Rate Risk, *Journal of International Money and Finance*, **14**, 667–678.

Broll, U. and K.P. Wong (1999) Hedging with Mismatched Currencies, *Journal of Futures Markets*, **19**, 859–875.

Broll, U., K.P. Wong and I. Zilcha (1999) Multiple Currencies and Hedging, *Economica*, **66**, 421–432.

Chang, E.C. and K.P. Wong (2003) Cross-Hedging with Currency Options and Futures, *Journal of Financial and Quantitative Analysis*, **38**, 555–574.

Eiteman, D.K., A.I. Stonehill and M.H. Moffett (2004) *Multinational Business Finance*, 10th ed. Addison-Wesley Publishing Company, Reading, MA.

Jacque, L.L. (1996) *Management and Control of Foreign Exchange Risk*. Kluwer Academic Publishers, Boston, MA.

Katz, E. and J. Paroush (1979) The Effect of Forward Markets on Exporting Firms, *Economics Letters*, **4**, 272–274.

Kawai, M. and I. Zilcha (1986) International Trade with Forward-Futures Markets under Exchange Rate and Price Uncertainty, *Journal of International Economics*, **20**, 83–98.

Lehmann, E.L. (1966) Some Concepts of Dependence, *Annals of Mathematical Statistics*, **37**, 1137–1153.

Shea, G. (1979) Monotone Regression and Covariance Structure, *Annals of Statistics*, **7**, 1121–1126.

Wong, K.P. (1996) Background Risk and the Theory of the Competitive Firm under Uncertainty, *Bulletin of Economic Research*, **48**, 241–251.

Wong, K.P. (2003) Currency Hedging with Options and Futures, *European Economic Review*, **47**, 833–839.

Wong, K.P. (2006) Optimal Export and Hedging Decisions When Forward Markets Are Incomplete, *Bulletin of Economic Research*, **58**, in press.