On the Investment-Uncertainty Relationship

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JEL classification: D21; D81; G13

Keywords: Real options; Investment timing; Uncertainty

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On the investment-uncertainty relationship

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Abstract

This paper examines the investment-uncertainty relationship in a canonical real options model. We show that the critical lump-sum payoff of a project that triggers the exercise of the investment option exhibits a U-shaped pattern against the volatility of the project. This is driven by two opposing effects of an increase in the volatility of the project: (i) the usual positive effect on option value, and (ii) a negative effect on option value due to the upward adjustment in the discount rate. We further show that such a U-shaped pattern is inherited by the expected time to exercise the investment option. Thus, for relatively safe projects, greater uncertainty may in fact shorten the expected exercise time and thereby enhance investment. This is in sharp contrast to the negative investment-uncertainty relationship as commonly suggested in the extant literature.

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1. Introduction

The investment-uncertainty relationship has been extensively studied in the literature, which by and large dictates a negative sign to such a relationship (see Caballero, 1991; Leahy and Whited, 1996). The purpose of this paper is to shed more light on this relationship in a canonical real options model of McDonald and Siegel (1986) and Dixit and Pindyck (1994). Viewing investment opportunities as perpetual American call options, firms endogenously devise their investment timing so as to maximize the option values. The optimal decision

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rule is that a project should be undertaken at the first instant when the value of the project reaches a critical level (the investment trigger). To gauge the effect of uncertainty on investment, we follow Wong (2007) to use the the expected time to exercise the investment option. If this measure decreases (increases) with the volatility of the project, one can reasonably infer a positive (negative) sign of the investment-uncertainty relationship.

Following Sarkar (2000) and Wong (2007, 2008), we employ the single-factor intertemporal capital asset pricing model (CAPM) of Merton (1973a) to determine the risk-adjusted rate of return on the project. We show that the behavior of the investment trigger with respect to the volatility of the project is non-monotonic. When the volatility of the project goes up, the usual positive effect on option value (Merton, 1973b) makes waiting more beneficial. This lifts up the investment trigger. On the other hand, there is a negative effect on option value due to the upward adjustment of the discount rate in accord with the CAPM. This makes waiting more costly and thus pushes down the investment trigger. We show that the negative effect dominates (is dominated by) the positive effect for low (high) levels of uncertainty, thereby rendering a U-shaped pattern of the investment trigger against the volatility of the project.

We further show that the expected time of investment inherits the U-shaped pattern of the investment trigger against the volatility of the project. Specifically, the positive effect on option value that calls for shortening the investment time dominates for relatively safe projects, while the negative effect on option value that calls for lengthening the investment time dominates for sufficiently risky projects. Thus, it is quite possible that greater uncertainty may in fact lure firms into making more investment through shortening the expected time to exercise the investment option, especially when projects are relatively safe. This is in sharp contrast to the negative investment-uncertainty relationship commonly found in the extant literature.

The rest of this paper is organized as follows. The next section delineates the real options model. Section 3 derives the investment trigger and the value of the investment option.

Sections 4 and 5 examine how the investment trigger and the expected time to exercise the investment option respond to an increase in the volatility of the project, respectively. The final section concludes.

2. The model

Consider a canonical real options model *à la* McDonald and Siegel (1986) and Dixit and Pindyck (1994). Time, indexed by $t \geq 0$, is continuous and the horizon is infinite. Uncertainty is modeled by a complete probability space, $(\Omega, \mathcal{F}, P)$.

At time 0, a firm owns the right to invest in a project that can be irreversibly undertaken at some endogenously chosen time, $\tau \geq 0$. The investment time, τ, is uncertain ex ante. Investing in the project costs the firm $I > 0$, which is instantly paid at time τ. In return, the firm receives a lump-sum payoff, V_τ, so that the net payoff of the project is $V_\tau - I$ at time τ. We assume that the lump-sum payoff process, $\{V_t : t \geq 0\}$, is governed by the following geometric Brownian motion:

$$\mathrm{d}V_t = \alpha V_t \mathrm{d}t + \sigma V_t \mathrm{d}Z_t, \tag{1}$$

where $\{Z_t : t \geq 0\}$ is a standard Wiener process defined on $(\Omega, \mathcal{F}, P)$, and $\alpha > 0$ and $\sigma > 0$ are the positive drift rate (expected growth rate) and volatility (standard derivation) per unit of time, respectively. The initial value of the lump-sum payoff process, $V_0 > 0$, is known at time 0.

Investing in the project is analogous to exercising a perpetual American call option in that the firm has the right, but not the obligation, to invest at some future time to be optimally chosen by the firm. Let $F(V_t)$ be the value of the investment option at time $t \geq 0$. The firm optimally exercises the investment option at the investment time, τ, when the lump-sum payoff, V_τ, reaches a threshold level, V^*, from below at the first instant. We refer to V^* as the investment trigger.

Following Sarkar (2000) and Wong (2007, 2008), we assume that the underlying asset, i.e., the lump-sum payoff, for the investment option can be completely spanned by financial assets traded in the market, where risk-adjusted rates of return on financial assets are determined by the single-factor intertemporal capital asset pricing model (CAPM) of Merton (1973a). Let Y_t be the price of an asset or a portfolio of assets that is perfectly correlated with V_t. Denote by $\rho > 0$ as the correlation of Y_t with the market portfolio.[1] Then, the price process, $\{Y_t : t \geq 0\}$, evolves over time according to the following geometric Brownian motion:

$$\mathrm{d}Y_t = (r + \lambda\rho\sigma)Y_t\mathrm{d}t + \sigma Y_t\mathrm{d}Z_t, \tag{2}$$

where $r > 0$ is the constant instantaneous riskless rate of interest, $\lambda > 0$ is the constant market price of risk per unit of time, and $r + \lambda\rho\sigma$ is the risk-adjusted rate of return on Y_t according to the CAPM. Let $\delta = r + \lambda\rho\sigma - \alpha > 0$ be the convenience yield or return shortfall on V_t.[2]

3. Solution to the model

For all $V_t \geq V^*$, the investment option is immediately exercised so that $F(V_t) = V_t - I$. On the other hand, for all $V_t < V^*$, the firm keeps the investment option alive. To derive $F(V_t)$ in this case, we construct the following dynamic portfolio: (i) Hold the investment option that is worth $F(V_t)$, and (ii) go short n units of the asset or portfolio of assets that completely spans V_t. The value of this dynamic portfolio is $F(V_t) - nY_t$. The total return

[1] Since most assets would have values that are positively correlated with the market portfolio, we do not consider the case that $\rho \leq 0$.

[2] The assumption that $\delta > 0$ is called for to ensure a finite investment trigger at the optimum. See Eq. (14).

from holding the portfolio over a short time interval, $\mathrm{d}t$, is given by

$$\begin{aligned}
&\mathrm{d}F(V_t) - n\mathrm{d}Y_t \\
&= F'(V_t)\mathrm{d}V_t + \frac{1}{2}F''(V_t)(\mathrm{d}V_t)^2 - n[(r+\lambda\rho\sigma)Y_t\mathrm{d}t + \sigma Y_t\mathrm{d}Z_t] \\
&= \left[\alpha V_t F'(V_t) + \frac{1}{2}\sigma^2 V_t^2 F''(V_t) - n(r+\lambda\rho\sigma)Y_t\right]\mathrm{d}t + [V_t F'(V_t) - nY_t]\sigma\mathrm{d}Z_t, \qquad (3)
\end{aligned}$$

where the first equality follows from Ito's Lemma and Eq. (2), and the second equality follows from Eq. (1) and $(\mathrm{d}V_t)^2 = \sigma^2 V_t^2\mathrm{d}t$. Substituting $n = V_t F'(V_t)/Y_t$ into Eq. (3) yields

$$\mathrm{d}F(V_t) - \frac{V_t F'(V_t)}{Y_t}\mathrm{d}Y_t = \left[\frac{1}{2}\sigma^2 V_t^2 F''(V_t) - \delta V_t F'(V_t)\right]\mathrm{d}t, \qquad (4)$$

where $\delta = r + \lambda\rho\sigma - \alpha$. Inspection of Eq. (4) reveals that the portfolio with $n = V_t F'(V_t)/Y_t$ is riskless and thus we must have

$$\left[\frac{1}{2}\sigma^2 V_t^2 F''(V_t) - \delta V_t F'(V_t)\right]\mathrm{d}t = r[F(V_t) - V_t F'(V_t)]\mathrm{d}t, \qquad (5)$$

to rule out arbitrage opportunities. Eliminating $\mathrm{d}t$ on both sides of Eq. (5) and rearranging terms yields

$$\frac{1}{2}\sigma^2 V_t^2 F''(V_t) + (r-\delta)V_t F'(V_t) - rF(V_t) = 0. \qquad (6)$$

Thus, $F(V_t)$ must satisfy Eq. (6) for all $V_t \in (0, V^*)$, where V^* is a free boundary to be optimally chosen by the firm.

Eq. (6) is a second-order linear homogeneous ordinary differential equation. The general solution to Eq. (6) is the sum of two powers:

$$F(V) = A_1 V^{\beta_1} + A_2 V^{\beta_2}, \qquad (7)$$

where A_1 and A_2 are constants to be determined, and β_1 and β_2 are the positive and negative roots, respectively, for the following fundamental quadratic equation:[3]

$$\frac{1}{2}\sigma^2\beta(\beta-1)+(r-\delta)\beta-r=0. \tag{8}$$

The two constants, A_1 and A_2, and the investment trigger, V^*, are determined using considerations that apply at the boundaries of the region, $(0, V^*)$. There are three boundary conditions:

$$F(0)=0, \tag{9}$$

$$F(V^*)=V^*-I, \tag{10}$$

and

$$F'(V^*)=1. \tag{11}$$

The first boundary condition, Eq. (9), simply reflects the fact that zero is an absorbing barrier for the geometric Brownian motion defined in Eq. (1). The second boundary condition, Eq. (10), is the value-matching condition such that the value of the investment option is equal to the net payoff of the project at the investment time, τ. The third boundary condition, Eq. (11), is the smooth-pasting condition, or high-contact condition, such that the investment trigger, V^*, is the one that maximizes the value of the investment option.[4]

For Eq. (9) to hold, Eq. (7) implies that $A_2=0$. Thus, we can ignore the negative solution for β in Eq. (8) so that we can simply write $\beta_1=\beta$, where

$$\beta=\frac{1}{2}-\frac{r-\delta}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}. \tag{12}$$

[3]Substituting Eq. (7) into Eq. (6) and simplifying the resulting equation yields Eq. (8).

[4]Shackleton and Sødal (2005) show that smooth pasting implies rate of return equalization between the option and the levered position that results from exercise.

Note that Eq. (8) can be written as

$$(\beta - 1)\left(\frac{1}{2}\sigma^2\beta + r\right) = \delta\beta. \tag{13}$$

Since $\beta > 0$ and $\delta > 0$, it is evident from Eq. (13) that $\beta > 1$. Using the remaining two boundary conditions, Eqs. (10) and (11), we solve the investment trigger, V^*, and the value of the investment option at time 0, $F(V_0)$, in the following proposition.

Proposition 1. *The investment trigger, V^*, is given by*

$$V^* = \left(\frac{\beta}{\beta - 1}\right)I, \tag{14}$$

and the value of the investment option at time 0 is given by

$$F(V_0) = \begin{cases} (V^* - I)(V_0/V^*)^\beta & \text{if } V_0 < V^*, \\ V_0 - I & \text{if } V_0 \geq V^*, \end{cases} \tag{15}$$

where β is defined in Eq. (12). Furthermore, for all $V_0 < V^*$, $F(V_0)$ is strictly convex and greater than $V_0 - I$.

Proof. Using Eq. (7) with $A_2 = 0$ and $\beta_1 = \beta$ defined in Eq. (12), we can write Eqs. (10) and (11) as

$$A_1 V^{*\beta} = V^* - I, \tag{16}$$

and

$$A_1 \beta V^{*\beta - 1} = 1, \tag{17}$$

respectively. Multiplying β to Eq. (16) and V^* to Eq. (17) and subtracting the resulting equations yields Eq. (14). Substituting $A_2 = 0$, $\beta_1 = \beta$, and Eq. (16) into Eq. (7) yields Eq. (15).

Note that

$$\frac{\mathrm{d}}{\mathrm{d}V}\left(\frac{V-I}{V^{\beta}}\right) = \frac{\beta I-(\beta-1)V}{V^{\beta+1}} = \frac{(\beta-1)(V^{*}-V)}{V^{\beta+1}}, \tag{18}$$

where we have used Eq. (14). Thus, if $V_0 < V^*$, Eq. (18) implies that

$$\frac{V^{*}-I}{V^{*\beta}} > \frac{V_0-I}{V_0^{\beta}}. \tag{19}$$

Rewriting inequality (19) yields

$$F(V_0) = (V^{*}-I)\left(\frac{V_0}{V^{*}}\right)^{\beta} > V_0 - I. \tag{20}$$

Furthermore, for all $V_0 < V^*$, we have

$$F''(V_0) = \frac{\beta(\beta-1)(V^{*}-I)V_0^{\beta-2}}{V^{*\beta}} > 0, \tag{21}$$

since $\beta > 1$. Hence, Eqs. (20) and (21) imply that $F(V_0)$ is strictly convex and greater than $V_0 - I$ for all $V_0 < V^*$. $\square$

Since $\beta > 1$, Eq. (14) implies that $V^* > I$. That is, the firm finds it optimal to exercise the investment option only when the net payoff of the project is sufficiently positive. The term, $(V_0/V^*)^{\beta}$, in Eq. (15) can be interpreted as the stochastic discount factor that accounts for both the timing and the probability of one dollar received at the first instant when the investment trigger, V^*, is reached from below. If $V_0 \geq V^*$, the investment option is immediately exercised at time 0 so that $F(V_0) = V_0 - I$. Otherwise, the firm keeps the investment option alive until the investment trigger, V^*, is reached from below at the first instant. In this case, $F(V_0) > V_0 - I$.

Figure 1 depicts the value of the investment option at time 0, $F(V_0)$, as a function of the initial value of the lump-sum payoff, V_0.

(Insert Figure 1 here)

4. The trigger-uncertainty relationship

In this section, we examine the effect of uncertainty on the investment trigger, V^*. To this end, we follow Sarkar (2000) and Wong (2007, 2008) to refer to an increase in uncertainty as an increase in σ, taking all other parameters, r, λ, ρ, and α, as constants. In this case, the increase in σ has a systematic risk component that affects the convenience yield, δ. In accord with the CAPM, we have $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$. In contrast, McDonald and Siegel (1986) and Dixit and Pindyck (1994) consider another type of increased uncertainty in which the convenience yield, δ, is held fixed when σ varies, i.e., $\mathrm{d}\delta/\mathrm{d}\sigma = 0$. In this regard, the increase in σ has only an idiosyncratic risk component. While we follow the approach of Sarkar (2000) and Wong (2007, 2008), our analysis is readily extended to the case of McDonald and Siegel (1986) and Dixit and Pindyck (1994) by setting $\mathrm{d}\delta/\mathrm{d}\sigma = 0$.

Differentiating Eq. (8) with respect to σ yields

$$\frac{\mathrm{d}\beta}{\mathrm{d}\sigma} = \frac{\beta}{\sigma^2(\beta - 1/2) + r - \delta}\left[\frac{\mathrm{d}\delta}{\mathrm{d}\sigma} - \sigma(\beta - 1)\right] = \frac{2\beta^2}{\sigma^2\beta^2 + 2r}\left[\frac{\mathrm{d}\delta}{\mathrm{d}\sigma} - \sigma(\beta - 1)\right], \tag{22}$$

where the second equality follows from Eq. (8). Differentiating Eq. (14) with respect to σ yields

$$\frac{\mathrm{d}V^*}{\mathrm{d}\sigma} = -\frac{I}{(\beta - 1)^2}\frac{\mathrm{d}\beta}{\mathrm{d}\sigma} = \frac{2\beta V^*}{(\beta - 1)(\sigma^2\beta^2 + 2r)}\left[\sigma(\beta - 1) - \frac{\mathrm{d}\delta}{\mathrm{d}\sigma}\right], \tag{23}$$

where the second equality follows from Eqs. (14) and (22). If $\mathrm{d}\delta/\mathrm{d}\sigma = 0$, it is evident from Eq. (23) that $\mathrm{d}V^*/\mathrm{d}\sigma > 0$ for all $\sigma > 0$. However, if $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$, the trigger-uncertainty relationship is no longer monotonic, as is shown in the following proposition.

Proposition 2. *If* $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$, *there exists a unique point,* $\sigma^* \in (0, \sqrt{2\alpha})$, *defined by*

$$\sigma^* = \sqrt{\left(\frac{2r - 2\alpha + \lambda^2\rho^2}{2\lambda\rho}\right)^2 + 2\alpha} - \frac{2r - 2\alpha + \lambda^2\rho^2}{2\lambda\rho}, \tag{24}$$

such that $\mathrm{d}V^*/\mathrm{d}\sigma < (>)\ 0$ *for all* $\sigma < (>)\ \sigma^*$.

Proof. Eq. (23) implies that $\mathrm{d}V^*/\mathrm{d}\sigma$ has the same sign as that of $\sigma(\beta-1)-\lambda\rho$. Using Eq. (12) and $\delta = r+\lambda\rho\sigma-\alpha$, we have

$$\sigma(\beta-1)-\lambda\rho = \sqrt{\left(\frac{\sigma^2-2\alpha}{2\sigma}+\lambda\rho\right)^2+2r}-\frac{2\alpha+\sigma^2}{2\sigma}. \tag{25}$$

Note that

$$\left[\sqrt{\left(\frac{\sigma^2-2\alpha}{2\sigma}+\lambda\rho\right)^2+2r}-\frac{2\alpha+\sigma^2}{2\sigma}\right]\left[\sqrt{\left(\frac{\sigma^2-2\alpha}{2\sigma}+\lambda\rho\right)^2+2r}+\frac{2\alpha+\sigma^2}{2\sigma}\right]$$
$$= \left(\frac{\sigma^2-2\alpha}{2\sigma}+\lambda\rho\right)^2+2r-\left(\frac{2\alpha+\sigma^2}{2\sigma}\right)^2$$
$$= \frac{1}{\sigma}[\lambda\rho\sigma^2+(2r-2\alpha+\lambda^2\rho^2)\sigma-2\alpha\lambda\rho]. \tag{26}$$

Inspection of Eqs. (25) and (26) reveals that $\sigma(\beta-1)-\lambda\rho$ has the same sign as that of $\lambda\rho\sigma^2+(2r-2\alpha+\lambda^2\rho^2)\sigma-2\alpha\lambda\rho$, which is negative or positive depending on whether σ is lower or higher than σ^*, respectively, where σ^* is defined in Eq. (24). It is evident from Eq. (24) that $\sigma^* > 0$. Note that

$$\left(\frac{2r-2\alpha+\lambda^2\rho^2}{2\lambda\rho}\right)^2+2\alpha < \left(\frac{2r-2\alpha+\lambda^2\rho^2}{2\lambda\rho}+\sqrt{2\alpha}\right)^2. \tag{27}$$

Taking the square root on both side of inequality (27) and rearranging terms yields $\sigma^* < \sqrt{2\alpha}$. □

To see the intuition of Proposition 2, we write Eq. (23) as

$$\frac{\mathrm{d}V^*}{\mathrm{d}\sigma} = \frac{2\beta V^*\sigma}{\sigma^2\beta^2+2r} - \frac{2\beta V^*}{(\beta-1)(\sigma^2\beta^2+2r)}\frac{\mathrm{d}\delta}{\mathrm{d}\sigma}. \tag{28}$$

The first term on the right-hand side of Eq. (28) captures the usual positive effect due to the enhanced value of the investment option in response to an increase in σ (Merton, 1973b), holding the convenience yield, δ, fixed. The firm as such is induced to wait longer by lifting up the investment trigger. The second term on the right-hand side of Eq. (28) captures a negative effect that arises from the fact that $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$ in accord with the CAPM. The upward adjustment of the discount rate reduces the present value of the investment option. This makes waiting less attractive and thus the firm is induced to lower the investment trigger. The two effects act against each other. When there is relatively little uncertainty, it is evident from Eq. (28) that the positive effect is at best second order. The negative effect, on the other hand, is always first order because the risk-adjusted rate of return on the project is linear in σ in accord with the CAPM. When uncertainty becomes greater, the positive effect dominates the negative effect because the significance of the positive effect grows exponentially with σ while that of the negative effect grows only linearly with σ. This explains why V^* has a U-shaped pattern against σ with the unique minimum attained at σ^*.

5. The investment-uncertainty relationship

In this section, we examine the sign of the investment-uncertainty relationship in the context of our real options model. To have an interesting case, we assume that $V_0 < V^*$ so that the investment option is not immediately exercised at time 0.

Let $X_t = \ln V_t$. Using Ito's Lemma, Eq. (1) implies that X_t is governed by the following arithmetic Brownian motion:

$$\mathrm{d}X_t = \left(\alpha - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}Z_t. \tag{29}$$

Let $\phi(X,\tau)$ be the probability density function of the investment time, τ, at which the lump-sum payoff of the project reaches the investment trigger, V^*, from the initial value, V_0, at the first instant, and $X = \ln(V^*/V_0)$. Define $L(X,\theta)$ as the Laplace transform, or characteristic function, of $\phi(X,\tau)$:

$$L(X,\theta) = \int_0^\infty \exp(-\theta\tau)\phi(X,\tau)\ \mathrm{d}\tau. \tag{30}$$

We solve $\phi(X,\tau)$ and $L(X,\theta)$ in the following proposition.

Proposition 3. *The probability density function of the investment time, τ, at which the lump-sum payoff of the project reaches the investment trigger, V^*, from the initial value, V_0, at the first instant is given by*

$$\phi(X,\tau) = \frac{X}{\sigma\sqrt{2\pi\tau^3}} \exp\bigg\{ -\frac{1}{2\sigma^2\tau}\bigg[X - \Big(\alpha - \frac{\sigma^2}{2}\Big)\tau\bigg]^2\bigg\}, \tag{31}$$

and the Laplace transform of $\phi(X,\tau)$ is given by

$$L(X,\theta) = \exp\bigg\{ -\frac{X}{\sigma^2}\bigg[\sqrt{\Big(\alpha - \frac{\sigma^2}{2}\Big)^2 + 2\sigma^2\theta} - \Big(\alpha - \frac{\sigma^2}{2}\Big)\bigg]\bigg\}, \tag{32}$$

where $X = \ln(V^*/V_0)$ and $V_0 < V^*$.

Proof. We know that $\phi(X,\tau)$ must satisfy the forward Kolmogorov, or Fokker-Planck, equation of motion:

$$\frac{1}{2}\sigma^2\frac{\partial^2\phi(X,\tau)}{\partial X^2} - \Big(\alpha - \frac{\sigma^2}{2}\Big)\frac{\partial\phi(X,\tau)}{\partial X} - \frac{\partial\phi(X,\tau)}{\partial\tau} = 0. \tag{33}$$

Taking Laplace transforms of Eq. (33) term by term gives

$$\begin{aligned}&\frac{1}{2}\sigma^2\int_0^\infty \exp(-\theta\tau)\frac{\partial^2\phi(X,\tau)}{\partial X^2}\,\mathrm{d}\tau \\ &-\Big(\alpha - \frac{\sigma^2}{2}\Big)\int_0^\infty \exp(-\theta\tau)\frac{\partial\phi(X,\tau)}{\partial X}\,\mathrm{d}\tau - \int_0^\infty \exp(-\theta\tau)\frac{\partial\phi(X,\tau)}{\partial\tau}\,\mathrm{d}\tau = 0.\end{aligned} \tag{34}$$

Using Eq. (30), we can write Eq. (34) as

$$\frac{1}{2}\sigma^2\frac{\partial^2 L(X,\theta)}{\partial X^2} - \Big(\alpha - \frac{\sigma^2}{2}\Big)\frac{\partial L(X,\theta)}{\partial X} - \theta L(X,\theta) = 0, \tag{35}$$

where the last term on the left-hand side of Eq. (35) follows from integrating by parts. Eq. (35) is a second-order linear differential equation with constant coefficients. The general solution to Eq. (35) is the sum of two exponentials:

$$L(X,\theta) = B_1\exp[\gamma_1(\theta)X] + B_2\exp[\gamma_2(\theta)X], \tag{36}$$

where B_1 and B_2 are constants to be determined, and $\gamma_1(\theta)$ and $\gamma_2(\theta)$ are the positive and negative roots, respectively, for the following quadratic equation:[5]

$$\frac{1}{2}\sigma^2\gamma^2 - \left(\alpha - \frac{\sigma^2}{2}\right)\gamma - \theta = 0. \tag{37}$$

The two constants, B_1 and B_2, are determined using two appropriate boundary conditions. First, note that

$$L(X,\theta) \leq L(X,0) = \int_0^\infty \phi(X,\tau)\,\mathrm{d}\tau \leq 1, \tag{38}$$

for all $X > 0$. For Eq. (38) to hold, Eq. (36) implies that $B_1 = 0$ or else $L(\infty,\theta)$ would be unbounded. Second, at $X = 0$, i.e., at $V_0 = V^*$, investment occurs immediately. In this case, $\phi(0,\tau)$ becomes the Dirac delta function so that $L(0,\theta) = \exp(-\theta \times 0) = 1$. It then follows from Eq. (36) with $B_1 = 0$ that $B_2 = 1$. Solving Eq. (37) for $\gamma_2(\theta)$ yields

$$\gamma_2(\theta) = -\frac{1}{\sigma^2}\left[\sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\theta} - \left(\alpha - \frac{\sigma^2}{2}\right)\right]. \tag{39}$$

Substituting $B_1 = 0$, $B_2 = 1$, and Eq. (39) into Eq. (36) yields Eq. (32). Inversion of Eq. (32) yields Eq. (31). □

Following Wong (2007) and Gutiérrez (2007), we adopt the following two definitions to characterize the sign of the investment-uncertainty relationship.

Definition 1. *Given that the probability of eventual investment is equal to one, the sign of the investment-uncertainty relationship is said to be positive (negative) if, and only if, the expected time of investment decreases (increases) with an increase in uncertainty.*

Definition 2. *Given that the probability of eventual investment is less than one, the sign of the investment-uncertainty relationship is said to be positive (negative) if, and only if, the probability of eventual investment increases (decreases) with an increase in uncertainty.*

[5]Substituting Eq. (36) into Eq. (35) and simplifying the resulting equation yields Eq. (37).

The probability of eventual investment, Π, is given by

$$\Pi = \int_0^\infty \phi(X, \tau)\ \mathrm{d}\tau = L(X, 0). \tag{40}$$

where the second equality follows from Eq. (30). Eqs. (32) and (40) imply that $\Pi = 1$ if, and only if, $\sigma \leq \sqrt{2\alpha}$. For all $\sigma > \sqrt{2\alpha}$, Eqs. (32) and (40) imply that

$$\Pi = \exp\left[-\left(1 - \frac{2\alpha}{\sigma^2}\right)\ln\left(\frac{V^*}{V_0}\right)\right] = \left(\frac{V_0}{V^*}\right)^{1-2\alpha/\sigma^2}, \tag{41}$$

which is less than one. The expected time of investment, $\mathrm{E}(\tau)$, is given by

$$\mathrm{E}(\tau) = \int_0^\infty \tau\phi(X, \tau)\ \mathrm{d}\tau = -\frac{\partial L(X, \theta)}{\partial \theta}\bigg|_{\theta=0}, \tag{42}$$

where the second equality follows from Eq. (30). Using Eqs. (32) and (42), we have

$$\mathrm{E}(\tau) = \left(\alpha - \frac{\sigma^2}{2}\right)^{-1}\ln\left(\frac{V^*}{V_0}\right), \tag{43}$$

which is well-defined only when $\sigma < \sqrt{2\alpha}$, i.e., when $\Pi = 1$.

For all $\sigma < \sqrt{2\alpha}$, we differentiate Eq. (43) with respect to σ to yield

$$\frac{\mathrm{dE}(\tau)}{\mathrm{d}\sigma} = \left(\alpha - \frac{\sigma^2}{2}\right)^{-2}\left[\sigma\ln\left(\frac{V^*}{V_0}\right) + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{V^*}\frac{\mathrm{d}V^*}{\mathrm{d}\sigma}\right]. \tag{44}$$

For all $\sigma > \sqrt{2\alpha}$, we differentiate Eq. (41) with respect to σ to yield

$$\frac{\mathrm{d}\Pi}{\mathrm{d}\sigma} = -\Pi\left[\frac{4\alpha}{\sigma^3}\ln\left(\frac{V^*}{V_0}\right) + \left(1 - \frac{2\alpha}{\sigma^2}\right)\frac{1}{V^*}\frac{\mathrm{d}V^*}{\mathrm{d}\sigma}\right]. \tag{45}$$

If $\mathrm{d}\delta/\mathrm{d}\sigma = 0$, we know from Eq. (23) that $\mathrm{d}V^*/\mathrm{d}\sigma > 0$ for all $\sigma > 0$. Eqs. (44) and (45) then imply that $\mathrm{dE}(\tau)/\mathrm{d}\sigma > 0$ and $\mathrm{d}\Pi/\mathrm{d}\sigma < 0$, thereby rendering a positive investment-uncertainty relationship according to Definitions 1 and 2. However, if $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$, the investment-uncertainty relationship becomes non-monotonic, as is shown in the following proposition.

Proposition 4. *If* $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$, *there exists a unique point,* $\sigma^{\mathrm{o}} \in (0, \sigma^*)$, *implicitly defined by*

$$\left.\frac{\mathrm{dE}(\tau)}{\mathrm{d}\sigma}\right|_{\sigma=\sigma^{\mathrm{o}}} = 0, \tag{46}$$

such that $\mathrm{dE}(\tau)/\mathrm{d}\sigma < 0$ *for all* $\sigma < \sigma^{\mathrm{o}}$, $\mathrm{dE}(\tau)/\mathrm{d}\sigma > 0$ *for all* $\sigma \in (\sigma^{\mathrm{o}}, \sqrt{2\alpha})$, *and* $\mathrm{d}\Pi/\mathrm{d}\sigma < 0$ *for all* $\sigma > \sqrt{2\alpha}$.

Proof. From Proposition 2, we know that $\mathrm{d}V^*/\mathrm{d}\sigma > 0$ for all $\sigma > \sigma^*$. Thus, Eq. (44) implies that $\mathrm{dE}(\tau)/\mathrm{d}\sigma > 0$ for all $\sigma \in [\sigma^*, \sqrt{2\alpha})$ and Eq. (45) implies that $\mathrm{d}\Pi/\mathrm{d}\sigma < 0$ for all $\sigma > \sqrt{2\alpha}$. Define the expression inside the squared brackets on the right-hand side of Eq. (44) as M:

$$M = \sigma \ln\left(\frac{V^*}{V_0}\right) + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{V^*}\frac{\mathrm{d}V^*}{\mathrm{d}\sigma}. \tag{47}$$

Differentiating Eq. (47) with respect to σ yields

$$\frac{\mathrm{d}M}{\mathrm{d}\sigma} = \ln\left(\frac{V^*}{V_0}\right) + \left(\alpha - \frac{\sigma^2}{2}\right)\left[\frac{1}{V^*}\frac{\mathrm{d}^2V^*}{\mathrm{d}\sigma^2} - \left(\frac{1}{V^*}\frac{\mathrm{d}V^*}{\mathrm{d}\sigma}\right)^2\right]. \tag{48}$$

Differentiating Eq. (23) with respect to σ yields

$$\frac{\mathrm{d}^2V^*}{\mathrm{d}\sigma^2} = \frac{2I}{(\beta-1)^3}\left(\frac{\mathrm{d}\beta}{\mathrm{d}\sigma}\right)^2 - \frac{I}{(\beta-1)^2}\frac{\mathrm{d}^2\beta}{\mathrm{d}\sigma^2}. \tag{49}$$

Substituting Eqs. (14), (23), and (49) into Eq. (48) yields

$$\frac{\mathrm{d}M}{\mathrm{d}\sigma} = \ln\left(\frac{V^*}{V_0}\right) + \left(\alpha - \frac{\sigma^2}{2}\right)\left[\frac{2\beta-1}{\beta^2(\beta-1)^2}\left(\frac{\mathrm{d}\beta}{\mathrm{d}\sigma}\right)^2 - \frac{1}{\beta(\beta-1)}\frac{\mathrm{d}^2\beta}{\mathrm{d}\sigma^2}\right]. \tag{50}$$

Differentiating Eq. (22) with respect to σ yields

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}\sigma^2} = \frac{8\beta^3[\lambda\rho - \sigma(\beta-1)][\lambda\rho - \sigma(2\beta-1)]}{(\sigma^2\beta^2+2r)^2}$$
$$-\frac{2\beta^2(\beta-1)}{\sigma^2\beta^2+2r} - \frac{8\beta^5\sigma^2[\lambda\rho-\sigma(\beta-1)]^2}{(\sigma^2\beta^2+2r)^3}. \tag{51}$$

Substituting Eqs. (22) and (51) into Eq. (50) yields

$$\frac{\mathrm{d}M}{\mathrm{d}\sigma} = \ln\left(\frac{V^*}{V_0}\right) + \left(\alpha - \frac{\sigma^2}{2}\right)\left\{\frac{4\beta^2[\lambda\rho - \sigma(\beta-1)]^2}{(\beta-1)^2(\sigma^2\beta^2+2r)^2} + \frac{8\beta^3\sigma[\lambda\rho - \sigma(\beta-1)]}{(\beta-1)(\sigma^2\beta^2+2r)^2} + \frac{2\beta}{\sigma^2\beta^2+2r} + \frac{8\beta^4\sigma^2[\lambda\rho - \sigma(\beta-1)]^2}{(\beta-1)(\sigma^2\beta^2+2r)^3}\right\}, \tag{52}$$

which is unambiguously positive for all $\sigma \in (0, \sigma^*)$ since in this case we have $\sigma(\beta-1)-\lambda\rho < 0$. It then follows from Eq. (44) that $\mathrm{dE}(\tau)/\mathrm{d}\sigma$ is strictly increasing in σ for all $\sigma \in (0, \sigma^*)$. Eq. (8) implies that $\beta \to r/\alpha$ as $\sigma \to 0$. Taking limit on both sides of Eq. (44) as $\sigma \to 0$ therefore yields

$$\lim_{\sigma\to 0} \frac{\mathrm{dE}(\tau)}{\mathrm{d}\sigma} = -\frac{\lambda\rho}{\alpha(r-\alpha)} < 0. \tag{53}$$

Since $\mathrm{dE}(\tau)/\mathrm{d}\sigma$ is strictly increasing in σ for all $\sigma \in (0, \sigma^*)$ and $\mathrm{dE}(\tau)/\mathrm{d}\sigma > 0$ for all $\sigma \in [\sigma^*, \sqrt{2\alpha})$, we conclude from Eq. (53) that there exists a unique point, $\sigma^{\mathrm{o}} \in (0, \sigma^*)$, implicitly defined in Eq. (46), such that $\mathrm{dE}(\tau)/\mathrm{d}\sigma < 0$ for all $\sigma \in (0, \sigma^{\mathrm{o}})$ and $\mathrm{dE}(\tau)/\mathrm{d}\sigma > 0$ for all $\sigma \in (\sigma^{\mathrm{o}}, \sqrt{2\alpha})$. □

To see the intuition of Proposition 4, we use Eq. (28) to recast Eq. (44) as

$$\frac{\mathrm{dE}(\tau)}{\mathrm{d}\sigma} = \left[\left(\alpha - \frac{\sigma^2}{2}\right)^{-2}\ln\left(\frac{V^*}{V_0}\right) + \left(\alpha - \frac{\sigma^2}{2}\right)^{-1}\frac{2\beta}{\sigma^2\beta^2+2r}\right]\sigma - \left(\alpha - \frac{\sigma^2}{2}\right)^{-1}\frac{2\beta}{(\beta-1)(\sigma^2\beta^2+2r)}\frac{\mathrm{d}\delta}{\mathrm{d}\sigma}. \tag{54}$$

Inspection of Eq. (54) reveals two effects that govern the expected time of investment when the volatility of the project goes up. As in the previous section, the first term on the right-hand side of Eq. (54) captures the positive effect while the second term captures the negative effect. Proposition 4 states that greater uncertainty may in fact lure the firm into making more investment through shortening the expected time of investment, especially when the project is relatively safe (i.e., $\sigma < \sigma^{\mathrm{o}}$). When the project is sufficiently risky (i.e.,

$\sigma > \sigma^{o}$), the usual negative investment-uncertainty relationship as suggested in the extant literature prevails. This non-monotonic investment-uncertainty relationship is driven by the U-shaped pattern of the investment trigger against the volatility of the project. Specifically, the negative effect that calls for shortening the investment time dominates for relatively safe projects, while the positive effect that calls for lengthening the investment time dominates for sufficiently risky projects.

6. Conclusion

This paper has examined the investment-uncertainty relationship in a canonical real options model of McDonald and Siegel (1986) and Dixit and Pindyck (1994) with a caveat: Risk-adjusted rates of return on projects are determined by the single-factor intertemporal capital asset pricing model (CAPM) of Merton (1973a). We have shown that the critical lump-sum payoff of a project, i.e., the investment trigger, that triggers the exercise of the investment option exhibits a U-shaped pattern against the volatility of the project. This U-shaped pattern is driven by two opposing effects. When the volatility of the project goes up, the usual positive effect on option value (Merton, 1973b) makes waiting more beneficial. This lifts up the investment trigger. On the other hand, there is a negative effect on option value due to the upward adjustment of the discount rate in accord with the CAPM. This makes waiting more costly and thus pushes down the investment trigger. We have shown that the negative effect dominates (is dominated by) the positive effect for low (high) levels of uncertainty. We have further shown that the U-shaped pattern of the investment trigger against the volatility is inherited by the investment-uncertainty relationship. For relatively safe projects, greater uncertainty may in fact shorten the expected time to exercise the investment option and thereby lure firms into making more investment, which is in sharp contrast to the negative investment-uncertainty relationship as commonly suggested in the extant literature.

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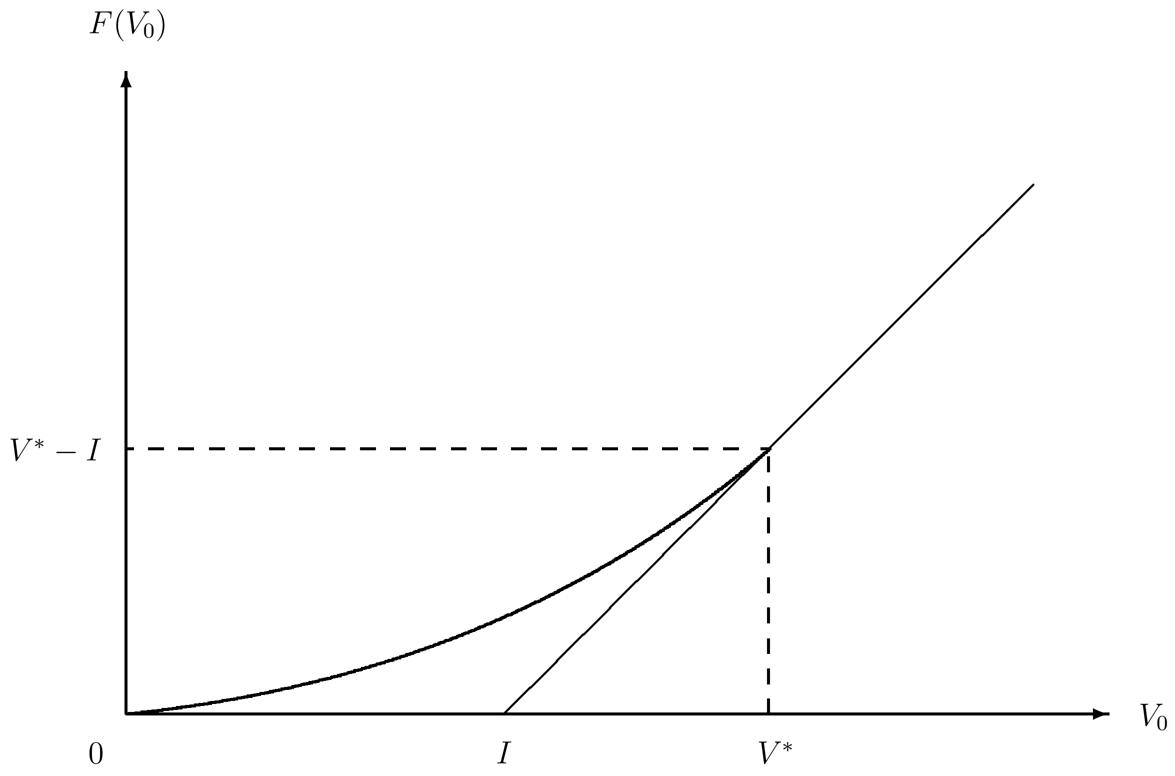

Figure 1